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UNIFORM BASES OF EXPONENTIALS, NEUTRAL GROUPS, AND A TRANSFORM THEORY FOR $H^m[a,b]$

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ABSTRACT

We present a theory relating the completeness and independence properties of sets of complex exponentials $\{e^{\lambda_k t}\}$ in the Sobolev spaces $H^m[a,b]$ to strongly continuous groups of bounded operators on $H^m[a,b]$ whose generator is the differentiation operator on the domain

$$\{x \in H^{m+1}[a,b] \mid \langle x,\eta\rangle = 0\}$$

where η is an element of $H^{m+1}[a,b]'$ whose Fourier transform has $\{\lambda_k\}$ as its zero set in the complex plane. In the process of proving our theorems we also develop a new approach to the use of the Laplace transform in the spaces $H^m[a,b]$.

AMS-MOS Subject Classification: 20M20, 44A10, 42A60, 42A96, 42A64.

Key Words: Exponential bases, Nonharmonic Fourier series, Semigroups of Operators, Laplace transform, Fourier Transform, Sobolev spaces

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SIGNIFICANCE AND EXPLANATION

In two previous MRC Technical Summary reports #1700, 2021 we have pointed out the relationship between bases of complex exponentials $\{e^{\lambda_k t}\}$ in the Sobolev spaces $H^m[a,b]$, neutral functional equations, and control canonical forms for systems governed by hyperbolic partial differential equations - the last mentioned being important in the development of stabilization and spectral assignment theories for such systems. These studies, and additional studies projected for related systems, have focussed our attention on the need to relate the classical theory of exponential bases to the more modern theory of semigroups of bounded operators. Attempts to carry this program out for $H^m[a,b]$ as well as $L^2[a,b]$, an important consideration for applications, have also resulted in the need for development of ways to apply the Laplace transform in $H^m[a,b]$, as presented in Section 3 ff. As a result we have obtained completeness and independence theorems for exponential bases in $H^m[a,b]$ in a simpler and more natural setting than in the existing literature.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

UNIFORM BASES OF EXPONENTIALS, NEUTRAL GROUPS,

AND A TRANSFORM THEORY FOR $H^m[a,b]$

DAVID L. RUSSELL

1. Introduction.

The central entities in this paper are generalized exponential functions, by which we mean the exponential functions, $e^{\lambda t}$, λ *complex*, t *real*, together with the functions $(t^k/k!)e^{\lambda t}$, $k = 1,2,3,\cdots$. For reference, and to fix the notation, we set down

Definition 1.1. A sequence of generalized exponentials is a set of functions

$$p_{j,k}(t) = (t^k/k!)e^{\lambda_j t} , \quad t \in I , \quad j \in J, \quad k = 0,1,2,\cdots,m_j-1 , \qquad (1.1)$$

where I is a closed real interval, J is a countable index set, the complex numbers λ_j , $j \in J$, are distinct, and the m_j, $j \in J$, are positive integers.

For purposes of abbreviation we may refer to the set of generalized exponentials associated with the exponents λ_j and the multiplicities m_j and denote the totality of such functions by $P(\lambda_j, m_j)$.

Let H be a complex Hilbert space whose elements are scalar functions defined on I or, possibly, distributions, in the sense of Schwartz [27], with support I . We need not specify the inner product $(\ , \ )$ and norm $\| \ \|$ at this point but if the reader thinks of $L^2[I]$ and the associated Sobolev spaces $H^r[I]$, r real, little will be lost (see [1], [6] for exposition of the properties of these spaces). We require that H should include all of the generalized exponentials $(t^k/k!)e^{\lambda t}$.

Given H and a sequence $P(\lambda_j,m_j)$ of generalized exponentials in H , we are primarily concerned with the completeness and independence properties of $P(\lambda_j, m_j)$ in H . We denote by $[P(\lambda_j, m_j)]$ the closed span of $P(\lambda_j,m_j)$ in H , i.e., the smallest closed subspace of H containing all finite linear combinations of elements of $[P(\lambda_j, m_j)]$. As usual, $P(\lambda_j, m_j)$ is complete in H if $[P(\lambda_j, m_j)] = H$. Independence is more complicated, as there are a number of varieties of this notion which must be considered.

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Definition 1.2. The sequence of generalized exponentials $P(\lambda_j, m_j)$ is

(i) weakly independent in H if the convergence of a series $\sum_{j \in J} \sum_{k=0} \alpha_{j,k} p_{j,k}$ to 0 in H implies all coefficients $\alpha_{j,k}$ are zero.

(ii) strongly independent if no p_{j_1,k_1} belongs to the closed subspace of H spanned by $p_{j,k}$, $(j,k) \neq (j_1,k_1)$;

(iii) uniformly independent if for any series $\sum_{j \in J} \sum_{k=1}^{m_j} \alpha_{j,k} p_{j,k}$ convergent to a limit $p \in H$ we have

$$d^{-2}\|p\|_H^2 \leq \sum_{j \in J} \sum_{k=0} |\alpha_{j,k}|^2 \leq D^2\|p\|_H^2 , \tag{1.2}$$

for some positive constants d,D.

If $P(\lambda_j, m_j)$ is complete in H and uniformly independent, we say that $P(\lambda_j, m_j)$ constitutes a uniform basis for H. This has been called the Riesz basis ([3], [18]). It is easy to see (cf [25], e.g.) that $P(\lambda_j, m_j)$ is a uniform basis for H if and only if, given an orthonormal basis B for H, there is a bounded and boundedly invertible operator $T : H \to H$ such that $T(B) = P(\lambda_j, m_j)$.

A generalization of the uniform basis concept is that of a uniform decomposition.

Definition 1.3 Let P_m, $m \in M$, where M is a countable index set, be a sequence of closed subspaces of H. These provide a uniform decomposition of H if given $p \in H$ there exist unique vectors $p_m \in P_m$ such that

$$p = \sum_{m \in M} p_m \tag{1.3}$$

and, for some positive constants d,D

$$d^{-2}\|p\|_H^2 \leq \sum_{m \in M} \|p_m\|_H^2 \leq D^2\|p\|_H^2 . \tag{1.4}$$

In the case $\dim P_m = 1$ for all m we may define $\hat{p}_m$ be the unique vector of norm 1 in P_m. Then we have $p_m = \alpha_m \hat{p}_m$ and

$$\begin{gathered} p = \sum_{m \in M} \alpha_m p_m \\ d^{-2}\|p\|_H^2 \leq \sum_{m \in M} |\alpha_m|^2 \leq D^2\|p\|_H^2 \end{gathered} \tag{1.5}$$

and we are back to a uniform basis (which, of course, need not consist of generalized exponentials as in (1.2). Clearly, vectors $\tilde{p}_m = \gamma_m \hat{p}_m$, $|\gamma_m|$, $|\gamma_m^{-1}|$ bounded, also form a uniform basis.

Given a uniform basis $P = p_m$ for H *there is defined, in a natural way, a dual* uniform basis $Q = \{q_m\}$ for H . If B is an orthonormal basis and $T(B) = P$, then

$$Q = (T^*)^{-1}(B).$$

Equivalently, $\{q_m\}$ is the unique element of H such that

$$(p_\ell, q_m)_H = \delta_{\ell m} = \begin{cases} 1, & \ell = m \\ 0, & \ell \neq m \end{cases} . \tag{1.6}$$

This notion extends to uniform decompositions with Q_m the orthogonal complement of the closed subspace spanned by the vectors in P_ℓ , $\ell \neq m$.

If $\{p_m\}$ is a uniform basis for h and $p \in H$, it is easy to see that p has the convergent development

$$p = \sum_{m \in M} \alpha_m p_m , \quad \alpha_m = (p, q_m)_H .$$

Slightly weaker properties are the following. If $P = \{p_m\}$ is strongly independent in H and each $p \in H$ has a unique series development

$$p = \sum_{m \in M} \alpha_m p_m \tag{1.7}$$

strongly convergent in H , we will say that P is a convergent basis for H . Correspondingly one can define strong decompositions of H , requiring the P_m to be strongly independent when $p_m \in P_m$ and the existence of a unique development (1.6).

It is customary in the case of Hilbert spaces to identify H with its dual space, but this is certainly not obligatory. Much of the work of J. L. Lions and his school is based on a different representation of the dual space H^* . See e.g. [15], [16]. In our applications of this procedure the "central" space is $H^0(I) = L^2[I]$. Suppose a second Hilbert space V , in practice usually one of the Sobolev spaces $H^r(I)$, or a variant thereof, is densely and continuously imbedded in $H^0(I)$, i.e. V may be identified with a subspace of $H^0(I)$ by means of a one to one and continuous injection map $J = V \to H^0(I)$. Thus $J : V \to J(V) \subseteq H^0(I)$, which we abuse slightly by writing $V \subset H^0(I)$. It is known then that there exists another Hilbert space, V', with $H^0(I) \subseteq V'$ such that V' is isometrically isomorphic to V . (See [H], [I] for details.) Further, a bilinear

form $\langle v,w \rangle$ may be defined for $v \in V$, $w \in V'$, equivalent to the inner product $(v,\bar{w})_{H^0(I)}$ when $w \in H^0(I)$, such that every continuous linear functional on V may be represented as

$$\ell(v) = \langle v,w \rangle$$

where w is a uniquely defined element of V'.

For our work in this paper the Hilbert space H containing the generalized exponentials $p_{j,k}(t)$ is always identified with V, In other applications H is identified with V', the dual space H' is identified with V (V is clearly reflexive, being a Hilbert space). If the $p_\ell(t)$ are taken to be in H, the $q_m(t)$ are taken to be in H' and (1.6) is replaced by

$$\langle p_\ell, q_m \rangle = \delta_{\ell,m} . \tag{1.8}$$

If the $p_\ell(t)$ lie in V', the $q_m(t)$ are in V and

$$\langle q_m, p_\ell \rangle = \delta_{\ell,m} . \tag{1.9}$$

Otherwise, everything remains as before.

The relationship between sets of generalized exponentials and group/semigroup theory arises very naturally. Given any finite sum

$$\sum_{j \in \hat{J}} \sum_{k=0}^{m_j-1} \alpha_{jk} \frac{t^k}{k!} e^{\lambda_j y} , \quad t \in I , \tag{1.10}$$

replacement of t by $t + T$ maps this sum into a new sum

$$\sum_{j \in \hat{J}} \sum_{k=0}^{m_j-1} \beta_{jk} \frac{t^k}{k!} e^{\lambda_j t} , \quad t \in I , \tag{1.11}$$

and the β_{jk} are readily computed from the $\alpha_{j,k}$. The inverse relationship arises from replacement of t by $t - T$ in (1.11).

Extending this relationship to infinite sums of generalized exponentials is not trivial: restrictions must be placed upon the λ_i and the m_j to realize the strongly continuous semigroup property and the strongly continuous group property requires even more. The central theme of this paper is the relationship between the uniform basis property of $P(\lambda_j, m_j)$ and the strongly continuous group property. This relationship is not yet complete in this paper but we are able to exhibit what we believe to be significant aspects of it. It will be seen that there is a broad correspondence between uniform bases of generalized exponentials for the Scbolev spaces $H^m[a,b]$ and scalar neutral functional

equations of the form

$$\zeta^{(m)}(t+b) = c_0\zeta^{(m)}(t+a) + \cdots$$

where $+ \cdots$ indicates lower order terms in a sense to be made precise later. But this class of equations is not adequate in itself and we are led to study neutral functional equations having a more general form which we discuss in the general framework of "neutral groups".

We are also interested in developing versions of the Laplace and Fourier transforms that are appropriate for use with the Sobolev spaces $H^m[a,b]$ on a finite interval $[a,b]$. It will be seen that there is a very natural way in which these transforms may be developed and that the resulting structure permits an esthetically pleasing and useful representation of these spaces which facilitates study of completeness and convergence properties of sets of generalized exponential functions.

2. Uniform Bases and Neutral groups.

We begin by defining the Sobolev spaces $H^m[a,b]$ for integer m. The space $H^0[a,b]$ is simply $L^2[a,b]$ with the usual inner product $(\ ,\ )$. For m a positive integer $H^m[a,b]$ consists of complex valued functions defined on $[a,b]$ possessing derivatives of order $\leq m$, in the sense permitted in the theory of distributions, which all lie in $L^2[a,b]$. The "standard" inner product in $H^m[a,b)$ is

$$\sum_{k=0}^{m} \int_a^b f^{(k)}(t)\overline{g^{(k)}(t)}dt \tag{2.1}$$

but this is not convenient for manipulations, as we have already explained in [J]. We define in $H^m[a,b]$ the inner product, equivalent to (2.1),

$$(f,g) = \sum_{k=0}^{m-1} f^{(k)}(c)\ \overline{g^{(k)}}(c) + \int_a^b f^{(m)}(t)\ \overline{g^{(m)}}(t)dt\ , \tag{2.2}$$

where c is an arbitrary, but fixed, point of $[a,b]$. The norm, of course, is

$$\|f\|_m = \sqrt{(f,f)}\ . \tag{2.3}$$

The dual space $H^m[a,b]'$ consists of distributions having the form, where $\delta_\alpha^{(k)}$ denotes the Dirac distribution of order k with support $\{\alpha\}$, $\alpha \in [a,b]$,

$$\sum_{k=0}^{m-1} \eta_k\ \delta_c^{(k)} + \eta_m \circledast \delta^{(k)} \tag{2.4}$$

The first part is self-explanatory. In the second part $\eta_m \in L^2[a,b]$ and, for $f \in H^m[a,b]$

$$\langle f,\ \eta_m \circledast \delta^{(k)}\rangle \equiv \int_a^b \eta_m(t)\ f^{(m)}(t)dt. \tag{2.5}$$

In the case where m is a negative integer it is best, to avoid confusion, to redefine m as positive and refer to the space as $H^{-m}[a,b]$. For the definition, we start with $H^{-m}[a,b]'$, which is $H_0^m[a,b]$,

$$H^{-m}[a,b]' \equiv H_0^m[a,b] = \{f \in H^m[a,b] f^{(k)}(a) = f^{(k)}(b) = 0\ ,\ \ k = 0,1,\cdots,m-1\}.$$

Then $H^{-m}[a,b]$ is the dual of $H^{-m}[a,b]'$ relative to $L^2[a,]$. It may be seen to be isometrically isomorphic to $H^m[a,b]'/B$ where B is the subspace of $H^m[a,b]'$ spanned by

$\delta_a^{(k)}$, $\delta_b^{(k)}$, $k = 0,1,2,\cdots,m-1$. Its properties are more fully explored in [22].

The theory of uniform bases of generalized exponential functions in $L^2[a,b]$ is approximately fifty years old and very well developed. Originating with Paley and Wiener [19], [31], it was initially concerned with the independence and spanning properties in $L^2[0,2\pi]$ of the sequences of exponential functions

$$e^{\lambda_k t}, \quad -\infty < k < \infty, \tag{2.6}$$

where the complex numbers λ_k were taken to be "close" to the imaginary integers ki. In some cases it was assumed that $\lambda_{-k} = -\lambda_k$. The major emphasis centered on the question: what is the largest positive number M_1 such that if

$$\sup_k |\lambda_k - ki| = M < M_1 \tag{2.7}$$

then (2.6) forms a uniform (or Riesz) basis for $L^2[0,2\pi]$? Increasingly large lower bounds for M_1 were established by Paley and Wiener [19], [31], Levinson [14]. Levinson's conjecture that $M_1 = 1/4$ was confirmed by Kadec [11] in 1964. From Levinson [14] it is known that one cannot take $M = M_1 = 1/4$.

Numerous generalizations of this basic result have been offered. It is easy to replace (2.7) by

$$\limsup_{k \to \infty} |\lambda_k - ki| = M < 1/4$$

retaining the uniform basis property. One first establishes strong independence, following that with an application of the Fredholm alternative. (See [23], e.g:) Any finite collection $e^{\lambda_{k_1} t}, e^{\lambda_{k_2} t}, \cdots, e^{\lambda_{k_n} t}$ can be replaced by the generalized exponentials $e^{\lambda_{k_1} t}, t e^{\lambda_{k_1} t}, \frac{t^2}{2} e^{\lambda_{k_1} t}, \cdots, t^{n-1}/(n-1)!\, e^{\lambda_{k_1} t}$ and this can be repeated any finite number of times. However, none of the intermediate powers of t may be skipped. One may select n complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_r$, all distinct modulo the integers, and set

$$\lambda_{k,j} = nki + \lambda_j + \varepsilon_{k,j}, \quad -\infty < k < \infty, \quad j = 1,2,\cdots,r,$$

and the $e^{\lambda_{k,j} t}$ will form a uniform basis for $L^2[0,2\pi]$ provided the $\lambda_{k,j}$ are all distinct and

$$\limsup_{\substack{k \to \infty \\ j=1,2,\cdots,r}} |\varepsilon_{k,j}| = M$$

with M suitably small. (See [5].) Again one may replace any finite number of strict exponentials $e^{\lambda_{k_1} t}, e^{\lambda_{k_1} t}, \cdots, e^{\lambda_{k_n} t}$ by a suitably sequence of generalized exponentials. One may allow duplication among $\lambda_1, \lambda_2, \cdots, \lambda_n$. The $\varepsilon_{k,j}$ must then be more severely restricted. (See Ullrich [30].) An excellent recent review has been offered by Redheffer [21]. A very general theory, related to our notion of a uniform decomposition of X into subspaces X_k as described in Section 1 is given in Schwartz [28].

All of these results trivally extend to $L^2[a,b]$ by replacing the "reference points" ki, $-\infty < k < \infty$, by $\left(\frac{2\pi}{b-a}\right)ki$, $-\infty < k < \infty$.

It has been known for sometime ([14], [28]) that if $P(\lambda_j, m_j)$ is a basis for $L^1[a,b]$ one may obtain a basis for $C[a,b]$ by adding at most one additional exponential function. *The converse is also true. If* $P(\lambda_j, m_j)$ *is a basis for* $C[a,b]$, then one may obtain a basis for $L^1[a,b]$ by removing at most one exponential. In this spirit a recent result [26], [22] establishes the following. If the exponentials $p_{j,k}(t) \in P(\lambda_j, m_j)$ form a uniform basis for $L^2[a,b]$, and if $\sigma_1, \sigma_2, \cdots, \sigma_m$ are m complex numbers, distinct and not equal to any of the λ_j, and if

$$p(\lambda) = \prod_{\ell=1}^{m} (\lambda - \sigma_\ell) , \tag{2.8}$$

then the functions

$$\left. \begin{array}{l} p_{j,k}(t)/p(\lambda_j), \quad -\infty < k < \infty , \quad j = 1,2,\cdots,m_j \\ \\ e^{\sigma_\ell t} , \ \ell = 1,2,\cdots,m \end{array} \right\}$$

provide a uniform basis for $H^m[a,b]$. In the other direction, if $\sigma_1, \sigma_2, \cdots, \sigma_m$ are m distinct complex numbers included in the λ_j and if $p(\lambda)$ is again defined by (2.8), the functions

$$p(\lambda_j) p_{j,k}(t), \quad -\infty < k < \infty , \quad j = 1,2,\cdots,m_j$$

form a uniform basis for $H^{-m}[a,b]$. Both results may be modified (see [22] for details) to include addition or deletion of generalized exponentials.

Having thus reviewed some of the important results regarding basis properties of generalized exponentials we may now proceed to the central theme of this paper, the connection between uniform bases of generalized exponentials in $H^m[a,b]$ and what we will call "neutral groups" in $H^m[a,b]$.

Definition 2.1. *Let* $S(t)$ *be a strongly continuous group of bounded operators on* $H^m[a,b]$, $m = 0,1,2,\cdots$. *We say that* $S(t)$ *is a neutral group in* $H^m[a,b]$ *if the generator of* $S(t)$ *is the differentiation operator*

$$\begin{aligned}(Ax)(t) &= (A(x(c), x'(c), \cdots, x^{(m-1)}(c), x^{(m)}(\cdot)))(t) \\ &= (x'(c), x''(c), \cdots, x^{(m)}(c), x^{(m+1)}(\cdot))\end{aligned} \tag{2.9}$$

defined on an appropriate domain $\mathcal{D}(A) \subseteq H^{m+1}[a,b] \subseteq H^m[a,b]$. *(We will show later that the domain of* A *necessarily takes the form*

$$\mathcal{D}(A) = \{x \subseteq H^{m+1}[a,b] \mid \langle x,\eta \rangle = 0\} \tag{2.10}$$

for a unique $\eta \in H^{m+1}[a,b]'$.)

An operator A *of the form* (2.9) *is a neutral generator if* A , *so defined, does indeed generate a strongly continuous group on* $H^m[a,b]$.

The "standard example" for η corresponds to what is usually called an m-th order neutral functional equation. The generator in that case being (2.9) with domain (2.10), where

$$\langle x,\eta \rangle = x^{(m)}(b) + \eta_0 x^{(m)}(a) + \sum_{k=0}^{m-1} \eta_{m-k} x^{(k)}(c) + \int_\alpha^b x^{(m)}(s)d\nu(s) , \tag{2.11}$$

ν being is a normalized function of bounded variation on $[a,b]$ such that, with V denoting the total variation

$$\lim_{\varepsilon\to 0} V(\nu,[a,a+\varepsilon]) = \lim_{\varepsilon\to 0} V(\nu,[b-\varepsilon,b]) = 0 .$$

See [7], [8], [17] for related existence and uniqueness theory. However, we will see in Section 5 that there are neutral generators on $H^m[a,b]$ which do not have this form. The construction involves the transform theory to be introduced in Section 3.

The two questions which interest us the most are the following.

Question A. Let $P(\lambda_j, m_j)$ be a uniform basis of generalized exponentials for $H^m[a,b]$. Does there exist a neutral generator A, as defined above, such that the generalized exponential solutions of

$$\frac{dx(t,s)}{dt} = A\,x(t,s), \quad -\infty < t < \infty, \quad s \in [a,b] \tag{2.12}$$

are precisely the generalized exponentials in $P(\lambda_j, m_j)$?

Question B. Let A be a neutral generator on $H^m[a,b]$ and let $P(\lambda_j, m_j)$ be the set of exponential solutions. Do the functions $P(\lambda_j, m_j)$ form a uniform, or perhaps just convergent, basis for $H^m[a,b]$?

By an exponential solution of (2.12) we mean, of course, a solution of the form

$$x(t,s) = ((t+s)^k/k!)e^{\lambda_j(t+s)}, \quad -\infty < t < \infty, \quad s \in [a,b]\ . \tag{2.13}$$

In studying these questions it is convenient to have the following lemma at our disposal, which allows us to study only neutral groups in $L^2[a,b]$.

Lemma 2.2. Let $p(\lambda) = \sum_{k=0}^{m} a_{m-k}\lambda^k$ be a complex polynomial of degree m and let

$$p(D) = \sum_{k=0}^{m} a_{m-k}D^k \quad (D = d/ds)$$

$$p(\delta_0') = \sum_{k=0}^{m} a_{m-k}\ \delta_0^{(k)}\ .$$

Let $\eta_0 \in H^1[a,b]'$ and let $(A_0x)(s) = x'(s)$ on the domain $\mathcal{D}(A_0 = \{x \in L^2[a,b] \mid x \in H^1[a,b],\ \langle x,\eta_0\rangle = 0\}$. Let $(A_m(x(c), x'(c), \cdots, x^{(m-1)}(c), x^{(m)}(\cdot)))(s) = (x'(c), x''(c), \cdots, x^{(m)}(c), x^{m+1}(\cdot))$ on the domain $\mathcal{D}(A_m) = \{x \in H^m[a,b] \mid x \in H^{m+1}[a,b],\ \langle x,\eta_m\rangle\} = 0$, where

$$\eta_m = p(\delta_0') * \eta_0 \in H^{m+1}[a,b]'\ . \tag{2.14}$$

Then A_0 is a neutral generator on $L^2[a,b]$ if and only if A_m is a neutral generator on $H^m[a,b]$.

Before giving the proof we note that the convolution product (2.14) of the distributions $p(\delta_0')$ and η_0 is defined as in [A]. In the proof itself we introduce the convolution product

$$\eta * \zeta$$

where $\eta \in H^{m+1}[a,b]'$ and ζ is defined on $(-\infty,\infty)$, $\zeta(t+\cdot) \in H^{m+1}[a,b]$ on $a \leq s \leq b$. As we will see in Section 4, the appropriate definition of $\eta_m * \zeta$ is

$$(\eta * \zeta)(t) = \langle \zeta(t+\cdot), \eta \rangle \text{ , a.e.} \tag{2.15}$$

On the right hand side of (2.15) we have the value of the linear functional $\eta \in H^{m+1}[a,b]'$ at the element $\zeta(t+\cdot) \in H^{m+1}[a,b)$, the $\langle \; , \; \rangle$ denoting that value, as is common in functional analysis.

Proof. Let us note that a linear map $P : E^m \oplus L^2[a,b] \to H^m[a,b]$ may be constructed as follows. Given

$$\xi = \begin{pmatrix} \xi^0 \\ \xi^1 \\ \vdots \\ \xi^{m-1} \end{pmatrix} \in E^m, \; \xi^m(\cdot) \in L^2[a,b] \tag{2.16}$$

$P(\xi) = \zeta = \tilde{\zeta} + \hat{\zeta} \in H^{m+1}_{\ell oc}(-\infty,\infty)$ where ζ is the unique solution of

$$P(D)\zeta = \xi^m \; , \; \zeta^{(k)}(c) = \xi^k \; , \quad k = 0,1,\cdots,m-1 \; .$$

The functions $\tilde{\zeta},\hat{\zeta}$ denote the solutions of $P(D)\tilde{\zeta} = 0$, $\tilde{\zeta}^{(k)}(c) = \xi^k$, $P(D)\hat{\zeta} = \xi^m$, $\hat{\zeta}^{(k)}(c) = 0$. Standard results from ordinary differential equations show that P: $E^m \oplus L^2[a,b] \to H^m[a,b]$ is one to one onto, bounded and boundedly invertible.

Suppose A_0 is a neutral generator on $L^2[a,b]$. Let $\xi_0^m \in L^2[a,b]$ and let $\xi^m(t,\cdot)$ be defined by

$$\xi^m(t,\cdot) = S_0(t)\xi_0^m, \quad t \in (-\infty,\infty) \; ,$$

where $S_0(t)$ is the group generated by A_0. When $\xi_0^m \in \mathcal{D}(A_0)$, $\xi^m(t,\cdot) \in \mathcal{D}(A_0)$ for all t so that

$$\langle \xi^m(t,\cdot), \eta_0 \rangle = 0 \; , \qquad t \in (-\infty,\infty) \; . \tag{2.17}$$

The equation

$$\frac{d\xi^m(t,\cdot)}{dt} = A_0\xi^m = (\xi^m)'(t,\cdot)$$

shows that ξ^m is, in reality, a function of $t+s$, so that

$$\xi^m(t+s) = \xi^m(t,s) \qquad s \in [a,b], \qquad -\infty < t < \infty$$

defines a function in $L^2_{\ell oc}(-\infty,\infty)$. When $\xi_0^m \in \mathcal{D}(A_0)$, $\xi^m \in H^1_{\ell oc}(-\infty,\infty)$ and (2.17) gives, in agreement with our earlier remarks

$$\eta_0 * \xi^m = 0 \ .$$

(We use the same symbol η_0 for the element of $H^1[a,b]'$ and the corresponding distribution with support in [a,b].)

Given $\xi \in E^m$ and $\xi_0^m \in L^2[a,b]$, the map

$$P(\xi,\xi_0^m) \to \zeta_0$$

associates with ξ,ξ_0 an element $\zeta_0 \in H^m[a,b]$. We have

$$\zeta_0 = \tilde{\zeta}_0 + \hat{\zeta}_1$$

as above. If we now let $\zeta(t)$ be the solution on $(-\infty,\infty)$ of

$$P(D)\zeta = \xi^m \tag{2.18}$$

with $\zeta^{(k)}(c) = \xi^k$, $k = 0,1,2,\cdots,m-1$, then $\zeta(t) = \zeta_0(t)$, $t \in [a,b]$, $\zeta^{(k)}(c) = \zeta_0{}^{(k)}(c)$, $k = 0,1,2,\cdots,m-1$. Since (2.18) is the same as

$$p(\delta') * \zeta = \xi^m$$

when $\xi_0^m \in \mathcal{D}(A_0)$ (just those cases wherein $\zeta_0 \in H^{m+1}[a,b]$ and $\langle \zeta_0, p(\delta') * \eta \rangle = 0$) we have in that case

$$p(\delta') * \eta * \zeta = \eta * p(\delta') * \zeta = \eta * \xi^m = 0 \ .$$

Letting $\zeta(t,\cdot)$ be the element of $H^m[a,b]$ represented by

$$\zeta(t+c),\ \zeta'(t+c),\ \cdots,\ \zeta^{(m-1)}(t+c),\ \zeta^{(m)}(t+\cdot))$$

we have, when $\zeta_0 \in H^{m+1}[a,b]$ and $\langle \zeta_0, p(\delta') * \eta \rangle = 0$,

$$\langle \zeta(t,\cdot),\ p(\delta') * \eta \rangle = 0$$

and

$$\frac{\partial\zeta(t,s)}{\partial t} = \frac{\partial\zeta(t,s)}{\partial s} \ .$$

When $\zeta \in H^{m+1}_{\ell oc}(-\infty,\infty)$,

$$\frac{\partial^{m+1}\zeta(t,s)}{\partial t(\partial s)^m} = \frac{\partial^{(m+1)}\zeta(t,s)}{(\partial s)^{m+1}} \ .$$

Moreover

$$\frac{d}{dt}\ \zeta^{(k)}(t,c) = \zeta^{(k+1)}(t,c), \quad k = 0,1,\cdots,m$$

so that

$$\frac{d}{dt}\,(\zeta(t,c),\zeta'(t,c),\cdots,\zeta^{(m-1)}(t,c),\zeta^{(m)}(t,\cdot)) =$$

$$(\zeta'(t,c),\ \zeta''(t,c),\ \cdots\ ,\ \zeta^{(m)}(t,c),\ \zeta^{(m+1)}(t,\cdot)) = A_m\zeta(t,\cdot)\ .$$

The strong continuity, bounded exponential growth etc of $\zeta(t,\cdot)$ in $H^m[a,b]$ can be inferred from the corresponding properties of $\xi^m(t,\cdot)$ and the familiar properties of solutions of $p(D)\zeta = \xi^m$. We omit the (decidedly unexciting) details.

The other direction is even simpler. Let ζ satisfy

$$\frac{d\zeta(t,\cdot)}{dt} = A_m\zeta(t,\cdot)\ .$$

For $\zeta_0 = \zeta(0,\cdot) \in \mathcal{D}(A_m)$ we have

$$p(\delta') * \eta_0 * \zeta = \eta_0\ p(\delta') * \zeta$$

so that, with $\xi = p(\delta') * \zeta, = p(D)\zeta$

$$\eta_0 * \xi = 0\ .$$

Since $\dfrac{\partial^{(k+1)}\zeta(t,s)}{\partial t(\partial s)^k} = \dfrac{\partial^{(k+1)}\zeta(t,s)}{(\partial s)^{k+1}}$, $k = 0,1,\cdots,m-1$, it is a simple matter to show that

$$\frac{\partial\xi(t,s)}{\partial t} = \frac{\partial\xi(t,s)}{\partial s}\ .$$

Thus ξ satisfies

$$\frac{d\xi(t,\cdot)}{dt} = A_0\xi(t,\cdot)\ .$$

Again the regularity properties are easily developed from those of ζ and the fact that $\zeta \to p(D)\zeta \equiv \xi$ is one to one, onto bounded and boundedly invertible on $H^m[a,b]/\ker p(D)$ to $L^2[a,b]$.

The proof of the next lemma is omitted being similar to the previous one and a result in [22],[26].

Lemma 2.3. Let $P(\lambda_j, m_j)$ be a collection of generalized exponentials, $p_{j,k}(t) = (t^k/k!)e^{\lambda_j t}$, $j \in J$, $k = 0,1,\cdots,m_j-1$. Let the index set J be expressed as a disjoint union

$$J = \bigcup_{\ell \in \Lambda} J_\ell$$

<u>and let</u> X_ℓ^0 <u>be the (closed) subspace of</u> $L^2[a,b]$ <u>spanned by</u> $p_{j,k}(t)$, $j \in J_\ell$, $k = 0,1,\cdots,m_j-1$.

<u>Let</u> $p(\lambda)$ <u>be a polynomial of degree</u> m <u>in</u> λ <u>and let</u> X_ℓ^m <u>be the subspaces of</u> $H^m[a,b]$ <u>spanned by the solutions</u> $z_{j,k}$ <u>of</u>

$$p(D)z_{j,k}(t) = p_{j,k}(t), \quad j \in J_\ell, \quad k = 0,1,\cdots,m_j-1 \tag{2.19}$$

<u>and let</u> Ξ^m <u>be the subspace of</u> $H^m[a,b]$ <u>spanned by the standard solutions of</u>

$$p(D)\ z(t) = 0\ . \tag{2.20}$$

<u>Then</u> Ξ^m, X_ℓ^m, $\ell \in \Lambda$, <u>form a uniform decomposition of</u> $H^m[a,b]$ <u>if and only if the</u> X_ℓ^0, $\ell = 1,2,3,\cdots$ <u>form a uniform decomposition of</u> $L^2[a,b]$.

<u>Remarks.</u> By "the standard solutions of $p(D)\ z_{j,k}(t) = p_{j,k}(t)$" we mean those particular solutions of this equation which are found by the method of undetermined coefficients. Thus if $p_{j,k}(t) = (t^k/k!)\ e^{\lambda_j t}$ and λ_j is a zero of $p(\lambda)$ of multiplicity m, the solution we are referring to is $\frac{1}{p^{(m)}(\lambda_j)} \left(\frac{t^{k+m}}{(k+m)!}\right) e^{\lambda_j t}$. We exclude addition of $p(D) = 0$. Thus, if $\tilde{X}_\ell^m$ is the subspace of all solutions of (2.20),

$$X_\ell^m = \tilde{X}_\ell^m / \Xi^m\ .$$

Alternatively, Lemma 2.3 can be restated as follows. If the X_ℓ^m, $\ell \in \tilde{\Lambda}$, are spanned by $p_{j,k}(t)$, $j \in \tilde{J}_\ell$, $k = 0,1,\cdots,m_j-1$, and if $\lambda_{j_1}, \lambda_{j_2}, \cdots, \lambda_{j_r}$, $m_{j_1}, m_{j_2}, \cdots, m_{j_r}$ are such that $\sum_{k=1}^{r} m_{j_k} = m$, $(t^q/q!)e^{\lambda_{j_k} t} \in \tilde{P}(\lambda_j, m_j)$, $q = 0,1,\cdots,m_{j_k}-1$, then the X_ℓ^m, $\ell \in \tilde{\Lambda}$, form a uniform decomposition of $H^m[a,b]$ if and only if the nontrivial spaces obtained from

$$X_\ell^0 = p(D)\ X_\ell^m$$

form a uniform decomposition of $L^2[a,b]$.

We can provide a non trivial partial answer to Question A immediately. Question B requires for its answer the transform theory which we will develop in Section 3.

Theorem 2.4. *Let* $P(\lambda_j, m_j)$ *be a collection of generalized exponentials as discussed earlier. Suppose the index set* J *can be divided into disjoint subsets* J_ℓ, $-\infty < \ell < \infty$, *with the following properties*

(i) *There exists a constant* $D > 0$ *such that for* $j, \hat{j} \in J_\ell$

$$|\operatorname{Im}(\lambda_j) - \operatorname{Im}(\lambda_{\hat{j}})| \leq 2D ;$$

(ii) *There is an integer* $N > 0$ *such that* J_ℓ *has at most* N *elements* λ_j, *including multiplicity, that is, for all* ℓ,

$$\sum_{j \in J_\ell} m_j \equiv N_\ell \leq N ;$$

(iii) *There is an* $M > 0$ *such that* $|\operatorname{Re}(\lambda_j)| \leq M$ *for all* $j \in J_\ell$;

(iv) *The spaces* X_ℓ *spanned by the generalized exponentials* $p_{j,k}(t)$, $j \in J_\ell$, $k = 0, 1, \cdots, m_j - 1$, *form a uniform decomposition of* $L^2[a,b]$. *Then there is an element* $\eta \in H^1[a,b]'$ *such that the* λ_j *are precisely the eigenvalues, with multiplicities* m_j, *of the operator*

$$(Ax)(s) = x'(s) ,$$

defined on the domain

$$\mathcal{D}(A) = \{x \in H^1[a,b] \mid \langle x, \eta \rangle = 0\} , \tag{2.21}$$

and A *is the generator of a neutral group* $S(t)$ *on* $L^2[a,b]$.

Proof. Let x_ℓ, $-\infty < \ell < \infty$, be arbitrary elements of X_ℓ and let

$$p_\ell(\lambda) = \prod_{j \in J_\ell} (\lambda - \lambda_j)^{m_j} = \sum_{r=0}^{N_\ell} a_{\ell,r} \lambda^{N_\ell - r} , \quad a_{\ell,0} = 1 , \tag{2.22}$$

be a monic polynomial of degree N_ℓ having the λ_j as zeros of multiplicity m_j. It is clear then that

$$p_\ell(D)\, x_\ell(t) = 0 , \qquad t \in [a,b] . \tag{2.23}$$

Moreover, $x_\ell(t)$ may be extended to t $(-\infty,\infty)$ by extending the solution of (2.23) from $[a,b]$ to $(-\infty,\infty)$. Since every $x \in L^2[a,b]$ can be written as

$$x = \sum_\ell x_\ell ,$$

$$C^{-2}\|x\|^2_{L^2[a,b]} \leq \sum_\ell \|x_\ell\|^2_{L^2[a,b]} \leq C^2\|x\|_{L^2[a,b]} ,$$

it is plausible to suppose that x can be extended from $[a,b]$ to $(-\infty,\infty)$ by extending the x_ℓ. But we need some estimates on the extended x_ℓ to establish this. Let

$w_\ell \in E^{N_\ell}$ be defined by

$$w_\ell(t) \equiv \begin{pmatrix} w_\ell^1(t) \\ w_\ell^2(t) \\ \vdots \\ w_\ell^{N_\ell}(t) \end{pmatrix} \equiv \begin{pmatrix} x_\ell(t) \\ x_\ell'(t) \\ \vdots \\ x_\ell^{(N_\ell-1)}(t) \end{pmatrix} .$$

Then

$$\dot{w}_\ell(t) = A_\ell w_\ell(t)$$

where A_ℓ is the $N_\ell \times N_\ell$ companion matrix

$$A_\ell = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{\ell,N_\ell} & -a_{\ell,N_\ell-1} & -a_{\ell,N_\ell-2} & \cdots & -a_{\ell,1} \end{pmatrix},$$

the $a_{\ell,r}$ being the coefficients in (2.22).

Then $x_\ell(t)$ is a linear functional of the vector $w_\ell(t)$; if $h^* = (1,0,0,\cdots,0)$,

$$x_\ell(t) = h^* w_\ell(t) .$$

In systems theory terminology, $x_\ell(t)$ is an "observation" on $w_\ell(t)$. A well known result (see, e.g. [24]) provides a formula giving $x_\ell(t)$, t arbitrary, in terms of $x_\ell(s)$, $s \in [a,b]$:

$$x_\ell(t) = h^* e^{A_\ell t} W_\ell^{-1} \int_a^b e^{A_\ell^* s} h\, x_\ell(s)ds, \tag{2.24}$$

where W_ℓ is the so-called "observability matrix"

$$W_\ell = \int_0^b e^{A_\ell^* s} h\, h^* e^{A_\ell s}\, ds . \tag{2.25}$$

Let t_0 be arbitrary. We claim that there is a positive number $M(t_0)$, independent of ℓ, such that

$$\|x_\ell\|_{L^2[t_0+a,t_0+b]} \leq M(t_0)\|x_\ell\|_{L^2[a,b]} . \tag{2.26}$$

To establish this we effect the transformation

$$y_\ell(t) = e^{-i\tau_\ell} x_\ell(t) , \tag{2.27}$$

where, from (i), τ_ℓ is a real number such that $|\tau_\ell - \mathrm{Im}(\lambda_j)| \leq D$, $j \in J_\ell$. The transformation (2.27) is an L^2 isometry from the subspace X_ℓ spanned by the $p_{j,k}(t)$, $j \in J_\ell$, $k = 0,1,\cdots,m_j-1$ to the subspace X_ℓ spanned by the functions

$$\tilde{p}_{j,k}(t) = (t^k/k!)e^{(\lambda_j - i\tau_\ell)t} ,$$
$$j \in J_\ell , \quad k = 0,1,\cdots,m_j-1 .$$

If we can prove an inequality like (2.26) for the $y_\ell(t)$, it is immediately established for $x_\ell(t)$. Let $z_\ell(t)$ correspond to $y_\ell(t)$ as $w_\ell(t)$ does to $x_\ell(t)$. Then

$$\dot{z}_\ell(t) = \tilde{A}_\ell z_\ell(t) ,$$
$$y_\ell(t) = h^* z_\ell(t) ,$$

where h^* is as before and $\tilde{A}_\ell$ is the companion matrix associated with the polynomial

$$\tilde{p}_\ell(\lambda) = \prod_{j \in J_\ell} (\lambda - (\lambda_j - i\tau_\ell))^{m_j} .$$

The formula (2.24) becomes

$$y_\ell(t) = h^* e^{\tilde{A}_\ell t} \tilde{W}_\ell^{-1} \int_a^b e^{\tilde{A}_\ell^* s} h\, y_\ell(s)ds. \tag{2.28}$$

From (i), (iii) in the statement of our theorem, the complex numbers $\tilde{\lambda}_j = \lambda_j - i\tau_\ell$, $j \in J_\ell$, satisfy the bound

$$|\tilde{\lambda}_j| \leq B = (D^2 + M^2)^{1/2}$$

with B independent of ℓ. Together with the inequality $N_\ell \leq N$, this implies a bound on the coefficients of $\tilde{p}_\ell$, hence a bound on the matrices $\tilde{A}_\ell$. We conclude that there is an M_0, independent of ℓ, such that

$$|h^* e^{\tilde{A}_\ell t} \tilde{W}_\ell^{-1} e^{\tilde{A}_\ell^* s} h| \leq M_0 e^{B|t|} , \tag{2.29}$$

provided only that the matrices

$$\tilde{W}_\ell = \int_a^b e^{\tilde{A}_\ell^* s} h h^* e^{\tilde{A}_\ell s} ds$$

have inverses $\tilde{W}_\ell^{-1}$ uniformly bounded independent of ℓ. But the pair $(h^*, \tilde{A}_\ell)$, $h^* = (1,0,\cdots,0)$, $\tilde{A}_\ell$ a companion matrix, is "observable" (see [24] again) in all cases which implies ([24]) that $\tilde{W}_\ell$ is invertible in all such cases. Since W_ℓ^{-1} is readily seen to be continuous with respect to the entries of $\tilde{A}_\ell$ and since those entries lie in a compact set in $\mathbf{C}^{N_\ell}$, $N_\ell \leq N$, we conclude that the $\tilde{W}_\ell^{-1}$ are bounded, independent of ℓ.

From the boundedness of the W_ℓ^{-1} a result (2.26) for the y_ℓ, and hence for the x_ℓ follows. Using (2.29) we can see, more specifically, that there are positive numbers M_0, B such that

$$\|x_\ell\|_{L^2[t_0+a,t_0+b]} \leq m_0 e^{B|t|} \|x_\ell\|_{L^2[a,b]} \tag{2.30}$$

Once we have (2.30) everything else follows quite quickly. Given $x \in L^2[a,b]$ we write

$$x(s) = \sum_\ell x_\ell(s)$$

with

$$C^{-2}\|x\|^2_{L^2[a,b]} \leq \sum_\ell \|x_\ell\|^2_{L^2[a,b]} \leq C^2\|x\|^2_{L^2[a,b]} . \tag{2.31}$$

Each x_ℓ is extended to $(-\infty,\infty)$ as a solution of (2.23). The result (2.30), along with the fact that the spaces X_ℓ are invariant, i.e.

$$x_\ell|_{L^2[a,b]} \in X_\ell \Longrightarrow x_\ell|_{L^2[t_0+a,t_0+b]} \in X_\ell ,$$

allows us to use the uniform decomposition property to see that $x \in L^2[t_0+a,t_0+b]$ for any t_0 and, in fact, there must be a constant $K(t_0) > 0$ such that

$$\|x\|_{L^2[t_0+a,t_0+b]} \leq K(t_0) e^{B|t_0|} \|x\|_{L^2[a,b]} .$$

We define the group $S(t)$ by

$$S(t): x|_{L^2[a,b]} \to x|_{L^2[t+a,t+b]} .$$

That $S(t)$ is bounded, uniformly on compact t-intervals is clear. Strong continuity follows from the fact that if we write

$$x(s) = \sum_{-L\le \ell \le L} x_\ell(s) + \sum_{|\ell|>L} x_\ell(s) \equiv \tilde{x}_L(s) + \hat{x}_L(s) \quad (\text{or } L^2[a,b] = \tilde{X}_L + \hat{X}_L)$$

then the maps

$$\tilde{S}_L(t) : \tilde{x}_L|_{L^2[a,b]} \to \tilde{x}_L|_{L^2[t+a,t+b]}$$

$$\hat{S}_L(t) : \hat{x}_L|_{L^2[a,b]} \to \hat{x}_L|_{L^2[t+a,t+b]}$$

are such that

$$S(t)x = \tilde{S}_L(t)\tilde{x}_L + \hat{S}_L(t)\hat{x}_L \;.$$

The operators $\tilde{S}_L(t)$ are, as functions of t, continuous in the uniform operator topology of $L(\tilde{X}_L, \hat{X}_L)$ while our estimates show $\hat{S}_L(t)$ uniformly bounded on compact intervals. Then, for $\tau \in I(t,\delta) = [t-\delta, t+\delta]$,

$$\|(S(t+\tau) - S(t))x\|^2 \le 2[\|(\tilde{S}_L(t+\tau) - \tilde{S}_L(t))\tilde{x}_L\|^2_{\tilde{X}_L} + \|(\hat{S}_L(t+\tau) - \hat{S}_L(t))\hat{x}_L\|^2_{\hat{X}_L}]$$

$$\le \|\tilde{S}_L(t+\tau) - \tilde{S}_L(t)\|^2 \|\tilde{x}_L\|^2_{\tilde{X}_L} + 2 \sup_{s \in I(t,\delta)} \|\hat{S}(s)\|^2 \|\hat{x}_L\|^2_{\hat{X}_L}$$

is easily used to establish that

$$\lim_{\tau \to t} \|(S(t+\tau) - S(t))x\| = 0 \;, \qquad x \in L^2[a,b].$$

There remains the question of identification of the generator of $S(t)$ and its domain. The existence of a closed generator A follows from the general theory [6],[12]. Further, it is clear, from the translation property of $S(t)$ that A must agree with the differentiation operator on the dense subspace

$$X_\infty = \bigcup_{L=1}^{\infty} \tilde{X}_L \;.$$

If we define

$$Bx = \sum_{-\infty < \ell < \infty} x'_\ell$$

with

$$\mathcal{D}(B) = \{x = \sum_{-\infty < \ell < \infty} x_\ell \mid x' \equiv \sum_{-\infty < \ell < \infty} x'_\ell \text{ converges in } L^2[a,b]\} \quad (2.32)$$

it is easy to verify that B is a closed extension of differentiation as defined on X_∞. Since A is likewise a closed extension of differentiation on X_∞, it is only necessary to show that these must agree. We begin by noting that the closedness of A together with $Ax_\ell = x'_\ell$, shows that $Ax = Bx$ for $x \in \mathcal{D}(B)$ as given in (2.32). Thus $A \supset B$. On the other hand, if $x \in \mathcal{D}(A)$ we can write

$$x = \sum_{-\infty < \ell < \infty} z_\ell , \quad z_\ell \in X_\ell ,$$

convergent in $L^2[a,b]$. Noting that A is invariant on each X_ℓ, A^* is invariant on the dual subspaces

$$Y_\ell = \Big[\sum_{k \neq \ell} X_k \Big]^\perp ,$$

each of which must belong to $\mathcal{D}(A^*)$. For $y_\ell \in Y_\ell$,

$$\begin{aligned} 0 &= (A^* y_\ell, x - x_\ell) = (y_\ell, Ax - Ax_\ell) \\ &= (y_\ell, \sum_k z_k - x'_\ell) = (y_\ell, z_\ell - x'_\ell). \end{aligned}$$

Since this is true for all $y_\ell \in Y_\ell$, we conclude $z_\ell - x'_\ell \in \sum_{k \neq \ell} X_k$. But $z_\ell - x'_\ell \in X_\ell$. This is only possible if $z_\ell = x'_\ell$ and we conclude that

$$x = \sum_{-\infty < \ell < \infty} z_\ell = \sum_{-\infty < \ell < \infty} x'_\ell$$

is convergent so that $x \in \mathcal{D}(B)$ and $Ax = Bx$. Thus $B \supset A$. It follows that $A = B$ on $\mathcal{D}(A) = \mathcal{D}(B)$ as given by (2.32).

Finally there is the matter of the form of $\mathcal{D}(A)$. Let σ be a complex number not included in the λ_j. Taking $m = 1$ in Lemma 2.3, it is then clear that X^1_ℓ and X^0_ℓ are subspaces of $H^1[a,b]$, $L^2[a,b]$ spanned by the same generalized exponentials. Since X^1_0, here spanned by $e^{\sigma t}$, together with the X^1_ℓ, form a uniform decomposition of $H^1[a,b]$, we conclude that there is a unique element $\eta \in H^1[a,b]'$ such that

$$\begin{aligned} \langle e^{\sigma \cdot}, \eta \rangle &= 1 \\ \langle x, \eta \rangle &= 0 , \quad x \in X^1_\ell , \quad -\infty < \ell < \infty . \end{aligned}$$

It is clear that $\{x \in H^1[a,b] \mid \langle x, \eta \rangle = 0\}$ is the closed span of the $p_{j,k}(t)$ in $H^1[a,b]$. It remains only to show that this is the same as $\mathcal{D}(A)$. This is clear, because

$$\{x \mid x = \sum_{-\infty < \ell < \infty} x_\ell \text{ convergent in } L^2[a,b] ,$$
$$x' \equiv \sum_{-\infty < \ell < \infty} x'_\ell \text{ convergent in } L^2[a,b]\}$$

is precisely $\{x \mid x = \sum_{-\infty < \ell < \infty} x_\ell$ convergent in $H^1[a,b]\}$ which is again the closed span of the $P_{j,k}(t)$ in $H^1[a,b]$. Thus

$$\mathcal{D}(A) = \{x \in H^1[a,b] \mid \langle x,\eta \rangle = 0\}$$

and the proof of Theorem 2.4 is complete.

<u>Corollary 2.5.</u> <u>Let Theorem</u> 2.4 <u>be reformulated with</u> $P(\lambda_j, m_j)$ <u>replaced by</u> $\tilde{P}(\lambda_j, m_j)$, J <u>by</u> $\tilde{J}$, J_ℓ <u>by</u> $\tilde{J}_\ell$, <u>and suppose</u> X_ℓ^m <u>is the subspace of</u> $H^m[a,b]$ <u>spanned by</u> $P_{j,k}(t)$, $j \in J_\ell$, $k = 0,1,\cdots,m_j-1$. <u>Let</u> (i), (ii), (iii) <u>hold with</u> (iv) <u>replaced by the assumption that the</u> X_ℓ^m <u>form a uniform decomposition for</u> $H^m[a,b]$. <u>Then there is an element</u> $\eta_m \in H^{m+1}[a,b]'$ <u>such that the</u> λ_j <u>are precisely the eigenvalues of the operator</u> A_m: $H^m[a,b] \rightarrow H^m[a,b]$ <u>defined by</u>

$$A_m(x(c),x'(c),\cdots,x^{(m-1)}(c),x^{(m)}(\cdot)) = (x'(c),x''(c),\cdots,x^{(m)}(c),x^{(m+1)}(\cdot)) \quad (2.33)$$

<u>on the domain</u>

$$\mathcal{D}(A) = \{x \in H^{m+1}[a,b] \subseteq H^m[a,b] \mid \langle x,\eta_m \rangle = 0\} , \quad (2.34)$$

<u>and</u> A <u>is the generator of a neutral group on</u> $H^m[a,b]$.

The proof may be sketched as follows. Let $\sigma_1,\sigma_2,\cdots,\sigma_m$ be any m distinct complex numbers among the λ_j and let

$$p(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k).$$

Define the sapces

$$X_\ell^0 = p(D)X_\ell^m .$$

These are spanned by subsets of the generalized exponentials spanning X_ℓ^m. Then use Theorem 2.4 to find

$$A_0 : L^2[a,b] \rightarrow L^2[a,b]$$

with domain

$$\mathcal{D}(A_0) = \{x \in H^1[a,b] \mid \langle x,\eta_0 \rangle = 0\}$$

for some $\eta_0 \in H^1[a,b]'$, A_0 generating a neutral group $S_0(t)$ on $L^2[a,b]$ and having

$$\{\lambda_j, j \in J\} = \{\lambda_j, j \in \tilde{J} | \lambda_j \neq \sigma_k, \ k = 1,2,\cdots,m\}$$

and its eigenvalues. Then, using Lemma 2.2, the operator A_m, defined by (2.33) as the domain (2.34) with

$$\eta_m = p(\delta_c') * \eta_0$$

has precisely the eigenvalues λ_j, $j \in \tilde{J}$ with multiplicities m_j and generates a neutral group $S_m(t)$ on $H^m[a,b]$. Further details are left to the reader.

It is also possible to state a similar Corollary for the spaces $H^{-m}[a,b]$, taking $P(\lambda_j, m_j)$ as in Theorem 2.4 and deleting m generalized exponentials from that set appropriately. The reader will have no trouble carrying this out if he is interested. The techniques required are similar to those used in [22].

There is, probably, a more straight-forward proof of Theorem 2.4. The essential difficulties occur as the λ_j, $j \in J_\ell$, are allowed to cluster together as j gets large.

Given A generating a neutral group $S(t)$ on $H^m[a,b]$, the element $\eta \in H^{m+1}[a,b]$ whose existence has been established in Theorem 2.4, will be called the generating functional, or generating distribution for $S(t)$.

In order to answer Question B ,and from intrinsic interest, we develop in Section 3 versions of the Fourier and Laplace transforms in $H^m[a,b]'$, $H^m[a,b]$, respectively, which are particularly adapted to the study of questions of this type. Additionally, we wish to set down the properties of these transformations here for use in later extensions of the work of this paper.

3. A Transform Theory for $H^m[a,b]'$, $H^m[a,b]$.

The classical treatments of nonharmonic Fourier series ([19], [14], [28], etc.) rely heavily on the Fourier transform for the development and proof of theorems. We have seen in Section 2, and earlier in [22], [26] that there are comparable theories for nonharmonic Fourier series in the Sobolev spaces $H^m[a,b]$. But how is this theory expressed in terms of transforms? The answer is not quite trivial, and we will see in this section that a combined use of the Fourier and Laplace transforms yields an elegant framework for the study of all sorts of questions of this type.

It is notationally simpler to take the point c in (2.2) to be zero. Since every interval $[a,b]$ which we are concerned with in this paper includes zero, this causes us no problems. Transformation of our results to other intervals is not difficult. For the work of the present section we specialize $[a,b]$ to $[-\pi,\pi]$, again no real restriction.

We have already introduced the Sobolev space $H^m[-\pi,\pi]$, representing elements (using $c = 0$ now) as

$$x = (x(0),\ x'(0),\cdots,x^{(m-1)}(0),x^{(m)}(\cdot)),\ x^m \in L^2[-\pi,\pi].$$

We choose to represent and think of $H^m[-\pi,\pi]'$ as a space of distributions y of the form

$$y = y_0\delta_0 + y_1\delta_0' + \cdots + y_{m-1}\delta_0^{(m-1)} + y_m \circledast \delta_0^{(m)}$$

where $y_0,y_1,\cdots,y_{m-1}$ are complex numbers, $\delta_0^{(r)}$ is the r-th distributional derivative of the Dirac distribution, δ_0, with support $\{0\}$ and $y_m \in L^2[-\pi,\pi]$. For $x \in H^m[-\pi,\pi]$, $y \in H^m[-\pi,\pi]$

$$\langle x,y\rangle = \sum_{k=0}^{m-1} y_k\, x^{(k)}(0) + \int_{-\pi}^{\pi} y_m(s)\, x^{(m)}(s)ds.$$

The norm of y is

$$\|y\|_{H^m[-\pi,\pi]'} = \left[\sum_{k=0}^{m-1} |y_k|^2 + \int_{-\pi}^{\pi} |y_m(s)|^2 ds\right]^{1/2}.$$

Definition 3.1. We define the Fourier transform F on $H^m[-\pi,\pi]'$ by

$$F(y,\lambda) = \langle p_\lambda, y \rangle, \quad p_\lambda(s) = e^{\lambda s}. \tag{3.1}$$

Thus

$$F(y,\lambda) = \sum_{k=0}^{m-1} y_k \lambda^k + \lambda^m \int_{-\pi}^{\pi} e^{\lambda s} y_m(s)ds. \tag{3.2}$$

The linear space of all (obviously entire) functions $\varphi(\lambda) = F(y,\lambda)$, $y \in H^m[-\pi,\pi]'$, will be referred to as $F(H^m[-\pi,\pi]') \equiv \Phi$, the latter designation being used when the space $H^m[-\pi,\pi]'$ is understood.

Theorem 3.2 The transforms $\varphi \in \Phi$ have the following properties $(\lambda = \xi + i\eta)$:

(i) $\varphi(\lambda)$ is entire;

(ii) for each real ξ, the function of η,

$$\varphi_\xi(\eta) \equiv \frac{\varphi(\xi + i\eta)}{(1 + |\xi + i\eta|)^m}, \in L^2(-\infty,\infty); \tag{3.3}$$

(iii) Φ is a Banach space (in fact a Hilbert space) with respect to each of the equivalent norms

$$\|\varphi\|_\rho = \left(\int_{Re(\lambda)=\rho} \left| \frac{\varphi(\lambda)}{(1 + |\lambda|)^m} \right|^2 |d\lambda| \right)^{1/2}, \ \rho \text{ real,}$$

and $F: H^m[-\pi,\pi]' \to \Phi (= F(H^m[-\pi,\pi)'))$ is an algebraic and topological isomorphism with respect to $\| \ \|_{H^m[-\pi,\pi]'}$ and $\| \ \|_\rho$.

Moreover, an entire function $\varphi \in \Phi$ if and only if

$$\varphi_\xi(\eta) \equiv \varphi(\xi + i\eta)/(1 + |\xi + i\eta|)^m \in L^2(-\infty,\infty) \tag{3.4}$$

for some real value of ξ and there is a positive M such that for all real ξ

$$\sup_{-\infty<\eta<\infty} |\varphi_\xi(\eta)| \leq \frac{Me^{\pi|\xi|}}{(1 + |\xi|)^{1/2}}. \tag{3.5}$$

Proof. Most of these properties are immediate consequences of corresponding theorems for F on $L^2[-\pi,\pi]$. In particular, (i) and (ii) are immediate from that theory and the form (3.2) of $\varphi(\lambda) = F(y,\lambda)$. With

$$\tilde{\varphi}(\lambda) = \int_{-\pi}^{\pi} e^{\lambda s} y_m(s)\,ds \tag{3.6}$$

the necessity of (3.5) for $\mathrm{Re}(\lambda) = \xi > 0$ follows from

$$\begin{aligned} \left|\frac{\tilde{\varphi}(\xi + i\eta)}{e^{\pi\xi}}\right| &= \left|\int_{-\pi}^{\pi} e^{\lambda s-\pi\xi} y_m(s)\,ds\right| \le \left(\int_{-\pi}^{\pi} |y_m(s)|^2 ds\right)^{1/2} \left(\int_{-\pi}^{\pi} e^{2\xi(s-\pi)}\,ds\right)^{1/2} \\ &\le \frac{M_0 \|y_m\|_{L^2[-\pi,\pi]}}{(1 + |\xi|)^{1/2}} \end{aligned} \tag{3.7}$$

where

$$M_0 = \sup_{\xi > 0} \left\{ \left|\frac{1 - e^{-4\pi\xi}}{2\xi}\right| (1 + |\xi|)^{1/2} \right\}$$

and a similar computation applies for $\xi < 0$. Setting $z_m(s) = e^{\xi s} y_m(s)$,

$$\tilde{\varphi}(\xi + i\eta) \quad (\text{cf. } (3.6)) = \int_{-\pi}^{\pi} e^{i\eta s} z_m(s)\,ds$$

and the Plancherel theorem gives

$$\frac{1}{2\pi} \|\tilde{\varphi}(\xi + i\cdot)\|^2_{L^2(-\infty,\infty)} = \|z_m\|^2_{L^2[-\pi,\pi]} \le e^{2\pi|\xi|} \|y_m\|^2_{L^2[-\pi,\pi]} ,$$

from which it is easy to see, since each of the functions $\lambda^k/(1 + |\lambda|)^m$, $k = 0,1,2,\cdots,m-1$, is square integrable on vertical lines, that there is a $B > 0$ such that (cf. (3.3))

$$\|\varphi_\xi\|_{L^2(-\infty,\infty)} \le B\, e^{\pi|\xi|} \|y\|_{H^m[-\pi,\pi]} . \tag{3.8}$$

This gives part of (iii) and establishes the necessity of (3.4).

For the sufficiency of (3.4), (3.5), we suppose $\varphi(\lambda)$ entire, satisfying these conditions. From Taylor's theorem, with $y_k = \varphi^{(k)}(0)/k!$, $k = 0,1,\cdots,m-1$,

$$\varphi(\lambda) = \sum_{k=0}^{m-1} y_k \lambda^k + \lambda^m \tilde{\varphi}(\lambda)$$

and (3.4) gives

$$\tilde{\varphi}(\xi + i\eta) = \frac{\varphi(\xi + i\eta) - \sum_{k=0}^{m-1} y_k(\xi + i\eta)^k}{(\xi + i\eta)^m} \in L^2(-\infty,\infty) \tag{3.9}$$

as a function of η. It follows that there exists $z_m \in L^2(-\infty,\infty)$ such that

$$\tilde{\varphi}(\xi + i\eta) = \underset{R \to \infty}{\text{l.i.m}} \int_{-R}^{R} e^{i\eta s} z_m(s)\,ds \tag{3.10}$$

$$z_m(s) = \frac{1}{2\pi} \underset{R \to \infty}{\text{l.i.m}} \int_{-R}^{R} e^{-i\eta s}\, \tilde{\varphi}(\xi + i\eta)\,d\eta \ .$$

Using (3.5) with a familiar result found in [14],[19], we see that the support of z_m is confined to $[-\infty,\infty]$. Setting

$$z_m(s) = e^{\xi s} y_m(s)$$

(3.10) becomes

$$\tilde{\varphi}(\xi + i\eta) = \int_{-\pi}^{\pi} e^{\lambda s}\, y_m(s)\,ds \tag{3.11}$$

and $\varphi(\lambda)$ is seen to have the form (3.2). This establishes the sufficiency of (3.4), (3.5).

The only interesting part of the work is the proof that $F^{-1} : \Phi \to H^m[-\pi,\pi]'$ is continuous. (The continuity of F is (3.8).) The proof we present here is based on a complex variables argument but has much in common with the proof of Lemma 2.2.

Let Γ_1 be the contour consisting of the two parallel lines $\mathrm{Re}(\lambda) = \rho$, $\mathrm{Re}(\lambda) = -\rho$, $\rho > 0$, the first oriented from $\eta = -\infty$ to $\eta = +\infty$, the second in the other direction. Let $p(\lambda)$ be a polynomial of degree m having no zeros on or in the interior of Γ_1. For $\varphi \in \Phi$,

$$\hat{\varphi}(\lambda) = \varphi(\lambda)/p(\lambda)$$

is holomorphic on and in the interior of Γ_1. Clearly

$$|\hat{\varphi}(\lambda)| \le B_0 \frac{|\varphi(\lambda)|}{(1 + |\lambda|)^m}, \lambda \in \Gamma_1 \cup \mathrm{Int}(\Gamma_1) \tag{3.12}$$

for some $B_0 > 0$. Elementary computations yield

$$\hat{\varphi}(0) = \frac{y_0}{p(0)}, \hat{\varphi}'(0) = \frac{y_1}{p(0)} - \frac{y_0 p'(0)}{p(0)^2}, \cdots \tag{3.13}$$

etc. From the Cauchy formula and (3.5)

$$\hat{\varphi}^{(k)}(0) = \frac{k!}{2\pi i} \int_{\Gamma_1} \frac{\hat{\varphi}(\lambda)\,d\lambda}{\lambda^{k+1}}, \ k = 0,1,2,\cdots,m-1 , \tag{3.14}$$

the convergence of the integrals following from (3.12) and (3.3). Since $1/\lambda^{k+1} \in L^2(\Gamma_1)$ for $k = 0,1,2,\cdots,m-1$, the Schwartz inequality gives

$$|\hat{\varphi}^k(0)| \leq B_k \|\hat{\varphi}\|_{L^2(\Gamma_\rho)} \leq B_0\tilde{B}_k(\|\varphi\|_\rho + \|\varphi\|_{-\rho}) ,$$

$k = 0,1,2, \cdots, m-1$, with $\tilde{B}_0,\tilde{B}_1,\cdots,\tilde{B}_{m-1}$ appropriate positive constants. From the triangular arrangement of the equations (3.13) and the equivalence of $\| \ \|_\rho$, $\| \ \|_{-\rho}$ it then follows that for some $B_k > 0$, $k = 0,1, \cdots, m-1$,

$$|y_k| \leq B_k \|\varphi\|_\rho , \ k = 0,1,\cdots,m-1 . \tag{3.15}$$

Next, let $\tilde{\varphi}(\lambda)$ be given by (3.9). Then from (3.15) and the fact that $|\lambda|^{k-m} \in L^2(\Gamma_1)$, $k = 0,1,\cdots,m-1$, we conclude that $\tilde{\varphi} \in L^2(\Gamma_1)$ and

$$\|\tilde{\varphi}\|_{L^2(\Gamma_1)} \leq b\|\varphi\|_\rho + \sum_{k=0}^{m-1} b_k|y_k|$$

for some positive constants b , b_k, $k = 0,1, \cdots, m-1$. Applying the Plancherel theorem to (3.11) on $Re(\lambda) = \rho$ and $Re(\lambda) = \rho$ then gives, for $\tilde{b} > 0$, $\tilde{b}_k > 0$, $k = 0,1,\cdots, m-1$,

$$\|y_m\|_{L^2[-\pi,\pi]} \leq \tilde{b}\|\varphi\|_\rho + \sum_{k=0}^{m-1} \tilde{b}_k \ |y_k| . \tag{3.16}$$

Combining (3.15) with (3.16) and our definition of $\| \ \|_{H^m[-\pi,\pi]'}$, we see that $F^{-1} = \Phi \to H^m[-\pi,\pi]'$ is continuous. The proof of Theorem 3.2 is now complete.

If the interval $[-\pi,\pi]$ in Theorem 3.2 is replaced by a general finite interval $[a,b]$, very little in regard to the space $H^m[a,b]'$ and $F(H^m[a,b]')$ changes. In fact, the only change is that (3.5) is replaced by

$$\sup_{-\infty<\eta<\infty} |\varphi_\xi(\eta)| \leq \begin{cases} \dfrac{Me^{b\xi}}{(1 + |\xi|)^{1/2}} , & \xi > 0 \qquad (3.5b) \\[2ex] \dfrac{Me^{a\xi}}{(1 + |\xi|)^{1/2}} , & \xi < 0 . \qquad (3.5a) \end{cases}$$

If A generates a neutral group $S(t)$ on $H^m[a,b]$ and $\eta \in H^{m+1}[a,b]'$ is the generating functional, the characteristic function for $S(t)$ (or A) is

$$\psi(\lambda) = F(\eta,\lambda) . \tag{3.17}$$

This agrees with the usual definitions. When η is given by (2.11) ,

$$\psi(\lambda) = \lambda^m e^{b\lambda} + \eta_0 \lambda^m e^{a\lambda} + \sum_{k=0}^{m-1} \eta_{m-k} \lambda^k e^{\lambda c}$$

$$+ \lambda^m \int_a^b e^{\lambda s} d\nu(s) \in H^{m+1}[a,b]'$$

and can easily be put in the standard form (3.2) with m replaced by $m+1$.

We turn now to a dual transform, or perhaps we should say a class of dual transforms for $H^m[-\pi,\pi]$ ' The transform theory for functions $x \in H^m[-\pi,\pi]$ is, in general, rather poorly developed. The problem lies in the fact that use of the classical Fourier-Laplace transform

$$\chi(\lambda) = \int_{-\pi}^{\pi} e^{-\lambda s} x(s)ds$$

does not lead to a transform $\chi(\lambda)$ whose properties readily reflect the differentiability of $x \in H^m[-\pi,\pi]$; the behavior of $\chi(i\eta)$ as $|\eta| \to \infty$ is dominated by the boundary terms associated with $s = a$, $s = b$. It is clear, of course, that one could trivially modify the F transform, as defined above for $H^m[-\pi,\pi]'$, and let

$$\chi(\lambda) = x(0) + \lambda x'(0) + \cdots + \lambda^{m-1} x^{(m-1)}(0) + \lambda^m \int_{-\pi}^{\pi} e^{\lambda s} x^{(m)}(s)ds$$

but this yields nothing new as compared with F on $H^m[-\pi,\pi]'$. It is not well suited to our work here.

The reader will recall that when the Fourier or Laplace transforms are used to solve a partial differential equation such as, e.g., $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ on a finite interval with certain boundary conditions applying at the endpoints of the interval, it is common to extend $y(x,t)$ to $-\infty < x < \infty$, where possible, by use of symmetry relations suggested by the boundary conditions. This is the essential idea of what we do to construct transforms of elements in $H^m[-\pi,\pi]$, the symmetry relations being replaced by the conditions that the function $x \in H^m[-\pi,\pi]$ be extended to $H^m_{\ell oc}(-\infty,\infty)$ as a solution of a neutral functional equation. Since there are infinitely many neutral functional equations which one could employ for this purpose, there are correspondingly infinitely many transforms. But we will see that this multiplicity of transforms serves well to treat the wide variety of possible expansions of x in terms of uniform bases of exponentials.

Let $S(t)$ be a neutral group on $H^m[-\pi,\pi]$ with generator A of the form (2.9). We make no assumption here about $\mathcal{D}(A) \subset H^{m+1}[-\pi,\pi]$ except that it be dense in that space and such that A is a neutral generator. This rules out, e.g., domains associated with

retarded equations such as

$$\mathcal{D}(A) = \{x \in H^{m+1}[-\pi,\pi] \,|\, x^{(m)}(\pi) = \sum_{k=0}^{m-1} a_k x^{(k)}(0) + \int_{-\pi}^{\pi} a(s)\, x^{(m)}(s)ds\}$$

since A generates only a semigroup with a domain of this type.

Given $x = x(s)$ in $H^m[-\pi,\pi]$, we define

$$x(t,s) = (S(t)x)(s). \tag{3.18}$$

By virtue of standard semigroup theory we know that $x(t,s)$ exists and belongs to $H^m[-\pi,\pi]$ for $-\infty < t < \infty$ and there are positive numbers M^+, M^-, real numbers γ^+, γ^- such that (cf. [6], [12])

$$\|x(t,\cdot)\|_{H^m[-\pi,\pi]} \leq \begin{cases} M^+ e^{\gamma^+ t}\|x\|_{H^m[-\pi,\pi]}, & t \geq 0 \\ M^- e^{\gamma^- t}\|x\|_{H^m[-\pi,\pi]}, & t \leq 0 \end{cases} \tag{3.19}$$

Further, we know that when $x \in \mathcal{D}(A) \subset H^{m+1}[-\pi,\pi]$, $x(t,\cdot) \in H^{m+1}[-\pi,\pi]$ for all t and

$$\frac{d\, x(t,\cdot)}{dt} = A\, x(t,\cdot) .$$

From the form of A, it is clear that, for $x \in \mathcal{D}(A)$,

$$x_A(t) \equiv x(t,0), \qquad -\infty < t < \infty \tag{3.20}$$

is such that x_A agrees with x on $[-\pi,\pi]$, and thus represents an extension of x from $[-\pi,\pi]$ to $(-\infty,\infty)$ determined by A - and hence by $\mathcal{D}(A)$ since the functional form of A itself is constant.

<u>Proposition 3.3.</u> <u>The function</u> $x_A(t)$ <u>has the following properties:</u>

(i) $x_A \in H^m[t_1,t_2]$ <u>on any finite interval</u> $[t_1,t_2]$, i.e., $x_A \in H^m_{loc}(-\infty,\infty)$;

(ii) <u>with</u> γ^+,γ^- <u>as in</u> (3.19), $e^{-\lambda t}x_A^{(k)}(t) \in L^p[0,\infty)$ <u>for</u> $Re(\lambda) > \gamma^+$, $\in L^p(-\infty,0]$ <u>if</u> $Re(\lambda) < \gamma^-$, $1 \leq p \leq \infty$, $k = 0,1,\cdots,m-1$;

(iii)
$$\|x_A\|_{H^m[-\pi+t,\pi+t]} \leq \begin{cases} M^+ e^{\gamma^+ t}\|x\|_{H^m[-\pi,\pi]}, & t \geq 0 \\ M^- e^{\gamma^- t}\|x\|_{H^m[-\pi,\pi]}, & t \leq 0 \end{cases} \tag{3.21}$$

<u>Proof.</u> All of these properties follow readily from the fact that, for $s \in [t-\pi,t+\pi]$

$$x_A(s) = x(t,s-t) = (S(t)x)(s-t) ,$$

and the estimates (3.19).

Definition 3.4. The L_A-transform of an element $x \in H^m[-\pi,\pi]$ is the Laplace transform of x_A, defined by

$$L_A(x,\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} x_A(t)\,dt, & \mathrm{Re}(\lambda) > \gamma^+ \\ -\int_{-\infty}^0 e^{-\lambda t} x_A(t)\,dt, & \mathrm{Re}(\lambda) < \gamma^- \end{cases} \tag{3.22}$$

and extended to other values of λ as described below.

Thereom 3.5 The operator A has compact resolvent $R(\lambda,A) = (\lambda I - A)^{-1}$, defined in $\rho(A)$ the resolvent set for A, which consists of the complement with respect to the complex plane of a countable set of points, $\sigma(A) \subseteq \{\lambda | \gamma^- < \mathrm{Re}(\lambda) < \gamma^+\}$. For each $x \in H^m[-\pi,\pi]$,

$$L_A(x,\lambda) = (R(\lambda,A)x)(0) \tag{3.23}$$

and is thereby extended, as compared with (3.22), to a meromorphic function of λ, analytic on $\rho(A)$.

Proof. We begin with (3.23). Since A generates a group $S(t)$ on $H^m[-\pi,\pi]$, for any complex λ the operator $A - \lambda I$ generates a group on $H^m[-\pi,\pi]$,

$$S(\lambda,t) = e^{-\lambda t} S(t) .$$

Now let $x \in \mathcal{D}(A)$. Then $S(\lambda,t)x \in \mathcal{D}(A) = \mathcal{D}(A - \lambda I)$ and

$$\frac{d}{dt} S(\lambda,t)x = (A - \lambda I)S(\lambda,t)x = S(\lambda,t)(A - \lambda I)x .$$

Using the fact that A, being a generator, is closed together with the convergence in $H^m[-\pi,\pi]$ of the integrals

$$\int_0^\infty S(\lambda,t)(A - \lambda I)x\,dt, \quad \mathrm{Re}(\lambda) > \gamma^+, \tag{3.24}$$

$$\int_0^\infty S(\lambda,t)(A - \lambda I)x\,dt, \quad \mathrm{Re}(\lambda) < \gamma^-, \tag{3.25}$$

we see that, working with the case $\mathrm{Re}(\lambda) > \gamma^+$,

$$\int_0^\infty (A - \lambda I)S(\lambda,t)x\,dt = \int_0^\infty S(\lambda,t)(A - \lambda I)x\,dt =$$

$$\int_0^\infty \frac{d}{dt} S(\lambda,t)x\,dt = \lim_{t\to\infty} S(\lambda,t)x - S(\lambda,0)x = -x .$$

Since this is true for all $x \in \mathcal{D}(A)$ and since A is closed, $S(\lambda,t)$ bounded, $\mathcal{D}(A)$ dense in $H^m[-\pi,\pi]$,

$$(A - \lambda I)\int_0^\infty S(\lambda,t)x\,dt = -x\;,\quad x \in H^m[-\pi,\pi]\;.$$

A similar computation applies to $\mathrm{Re}(\lambda) < \gamma^-$. Thus

$$\int_0^\infty S(\lambda,t)x\,dt = (\lambda I - A)^{-1}x,\quad x \in H^m[-\pi,\pi], \tag{3.26}$$

for $\mathrm{Re}(\lambda) > \gamma^+$ or $\mathrm{Re}(\lambda) < \gamma^-$. Now the range of A is contained in $H^{m+1}[-\pi,\pi]$, so we conclude that the integral on the left of (3.26) is in that space. Since $\delta_{(0)}$ is in $H^{m+1}[-\pi,\pi]'$

$$\langle \delta_{(0)}, \int_0^\infty S(\lambda,t)x\,dt\rangle = \langle \delta_{(0)}, (A - \lambda I)^{-1}x\rangle = ((A - \lambda I)^{-1}x)(0)\;.$$

It is easy to see that

$$\langle \delta_{(0)}, \int_0^\infty S(\lambda,t)x\,dt\rangle = \int_0^\infty (S(\lambda,t)x)(0)dt = \int_0^\infty e^{-\lambda t}\,x_A(t)dt = L_A(x,\lambda) \tag{3.27}$$

if $m \geq 1$, or for $x \in \mathcal{D}(A)$ when $m = 0$. But the last integral is continuous with respect to $\|x\|_{L^2[-\pi,\pi]}$ even in the latter case. Thus (3.23) is established for $\mathrm{Re}(\lambda) > \gamma^+$. A similar argument applies for $\mathrm{Re}(\lambda) < \gamma^-$.

That $R(\lambda,A)$ is holomorphic in $\rho(A)$ is a standard result, see e.g. [6]. In order to show that $(\lambda I - A)^{-1}$ is compact, we show that if

$$W = \{w \in H^m[-\pi,\pi]\,|\,w = (\lambda I - A)^{-1}z,\ z \in Z \text{ bounded in } H^m[-\pi,\pi]\}$$

then W is pre-compact in $H^m[-\pi,\pi]$. From the form (2.9) of A, $w = (\lambda I - A)^{-1}z$, or $(\lambda I - A)w = z$, gives

$$w^{(m+1)}(s) - \lambda w^{(m)}(s) + z^{(m)}(s) = 0\;,\quad s \in [-\pi,\pi]\;. \tag{3.28}$$

$$w^{(k+1)}(0) - \lambda w^{(k)}(0) + z^{(k)}(0) = 0\;,\quad k = 0,1,2,\cdots,m-1\;. \tag{3.29}$$

If $m = 0$, (3.29) is void.

For $\mathrm{Re}(\lambda) > \gamma^+$ we have, as we have seen

$$w(0) = ((\lambda I - A)^{-1}z)(0) = \int_0^\infty e^{-\lambda t}\,z_A(t)dt$$

and then (3.19) gives

$$\begin{aligned} |w(0)| &\leq M^+\|z\|_{H^m[-\pi,\pi]}\sum_{\ell=0}^\infty e^{-(2\ell-1)\pi\,\mathrm{Re}(\lambda)}\,e^{\gamma^+ 2\ell\pi} \\ &\equiv B_0(\lambda)\|z\|_{H^m[-\pi,\pi]}\;. \end{aligned} \tag{3.30}$$

We conclude that if $\|z\|_{H^m[-\pi,\pi]}$ is bounded, then $|w(0)|$ is bounded. Since $|z^k(0)| \le \|z\|_{H^m[-\pi,\pi]}$, $k = 0,1,2,\cdots,m-1$, (3.29) may be used successively for $k = 0,1,2,\cdots,m-1$ to see that

$$|w^{(k)}(0)| \le B_k(\lambda)\|z\|_{H^m[-\pi,\pi]}, \quad k = 0,1,2,\cdots m . \tag{3.31}$$

Combining (3.31) with the differential equation (3.28) for $w^{(m)}$ we see that the elements, w, of W are such that $w^{(k)}$, $k = 0,1,\cdots,m$ are pointwise bounded, $w^{(m+1)}$ is bounded in $L^2[-\pi,\pi]$, when Z is bounded and it follows that $R(\lambda,A)$ is compact for $Re(\lambda) > \gamma^+$. A similar argument applies to $Re(\lambda) < -\gamma^-$ and the "resolvent identity" (cf. [6])

$$((\lambda + \mu)I - A)^{-1} = (\lambda I - A)^{-1}(I + \mu(\lambda I - A)^{-1})$$

then shows $R(\lambda,A)$ to be compact for any $\lambda \in \rho(A)$. Familiar results on compact operators [6] shows that

$$((\lambda I - A)^{-1}x)(0) = (R(\lambda,A)x)(0)$$

is meromorphic, having a countable set of poles of finite multiplicity in $\gamma^- < Re(\lambda) < \gamma^+$ which have no finite accumulation point. Since we have seen that (3.23) is true for $Re(\lambda) > \gamma^+$, $Re(\lambda) < \gamma^-$, that formula may be used to extend the definition of $L_A(x,\lambda)$ to $\rho(A)$. The proof is now complete.

It is natural to ask if every neutral group on $H^m[a,b]$, having a generator of the form (2.9), has associated with it a unique generating functional $\eta \in H^{m+1}[a,b]'$ such that $\mathcal{D}(A)$ is the subset of $H^m[a,b]'$ given by (2.10). The answer to this is in the affirmative. We use $H^m[-\pi,\pi]$ for the proof here but the extension to other intervals is immediate.

<u>Theorem 3.6.</u> <u>Let</u> A, <u>having the form</u> (2.9), <u>be a neutral generator on</u> $H^m[-\pi,\pi]$. <u>Then there is a unique</u> $\eta \in H^{m+1}[-\pi,\pi]' - H^m[-\pi,\pi]'$ <u>such that</u> $\|\eta\|_{H^{m+1}[-\pi,\pi]'} = 1$ <u>and</u> $\mathcal{D}(A)$ <u>has the form</u> (2.10).

<u>Proof.</u> The proof is much the same as that given for Theorem 2.4, slightly abstracted here because we do not know, a priori, that we have any uniform decomposition to work with.

Let A be a neutral generator on $H^m[-\pi,\pi]$, let $x_A(t)$ denote the extension of $x \in H^m[-\pi,\pi]$ to $(-\infty,\infty)$ as described by (3.20) and let M^+, M^-, γ^+, γ^- be as in (3.21). Let us note that each $x \in H^{m+1}[-\pi,\pi]$ can be uniquely represented as

$$x = (x(0),\ x'(\cdot)),\quad x' \in H^m[-\pi,\pi]$$

and

$$\|x\|^2_{H^{m+1}[-\pi,\pi]} = |x(0)|^2 + \|x'\|^2_{H^m[-\pi,\pi]} .$$

Let a be a positive number greater than γ^+. Let

$$z(t) = x'(t) - ax(t),\quad t \in [-\pi,\pi] \tag{3.32}$$

$$z_A(t) = (x_A)'(t) - a\, x_A(t),\qquad -\infty < t < \infty . \tag{3.33}$$

Then $z \in H^m[-\pi,\pi]$. Let us define, for $t \in [-\pi,\pi]$,

$$\begin{aligned}\hat{x}(t) &= -\int_0^\infty (e^{(A - aI)s}\, e^{At}z)(0)ds \\ &= -L_A(e^{At}z,a) = ((A - aI)^{-1}e^{At}z)(0) \\ &= (e^{At}(A - aI)^{-1}z)(0) = ((A - aI)^{-1}z)(t)\end{aligned} \tag{3.34}$$

and let $\hat{x}_A(t)$ be defined similarly for $t \in (-\infty,\infty)$, extending $\hat{x}(t)$. The fact that $a > \gamma^+$ ensures the convergence of the integral and the boundedness of $(A - aI)^{-1}$.

Since

$$(A - aI)^{-1}\, e^{At}z = e^{At}[(A - aI)^{-1}z]$$

and $(A - aI)^{-1}z \in \mathcal{D}(A) \subseteq H^{m+1}[-\pi,\pi]$, we conclude that $\hat{x}(t)$ is strongly differentiable with respect to t and

$$\hat{x}'(t) - a\,\hat{x}(t) = (A - aI)\hat{x}(t) = (e^{At}z)(0) = z(t),\quad t \in [-\pi,\pi] \tag{3.35}$$

the relationship extending to

$$(\hat{x}_A)'(t) - a\,\hat{x}_A(t) = z_A(t),\qquad t \in (-\infty,\infty) . \tag{3.36}$$

Let

$$x = \tilde{x} + \hat{x} . \tag{3.37}$$

From (3.32), (3.35), (3.36), (3.37) it follows that

$$\tilde{x}'(t) - a\tilde{x}(t) = 0 \ , \quad t \in [-\pi,\pi], \quad (\tilde{x}_A)'(t) - a\,\tilde{x}_A(t) = 0 \quad t \in (\infty,\infty) \ .$$

It follows that

$$\tilde{x}(t) = e^{at}\,\tilde{x}(0) = e^{at}(x(0) - \hat{x}(0)), \quad t \in [-\pi,\pi] \ ,$$

$$\tilde{x}_A(t) = e^{at}\tilde{x}(0) = e^{at}(x(0) - \hat{x}(0)), \quad t \in (-\infty,\infty) \ .$$

We see therefore, that every $x \in H^m[-\pi,\pi]$ can be uniquely written as a sum (3.37) where $\hat{x}$ has the form (3.34) for $z = x' - ax \in H^m[-\pi,\pi]$ and

$$\tilde{x}(t) = e^{at}(x(0) - \hat{x}(0)) = e^{at}(x(0) - ((A - aI)^{-1}z)(0))$$

$$= e^{at}(x(0) - ((A - aI)^{-1}(x' - ax))(0)).$$

If we let $\tilde{X}$ be the one dimensional subspace of $H^{m+1}[-\pi,\pi]$ spanned by e^{at} and if we let $\hat{X}$ be the range of $(A - aI)^{-1}(D - aI) \equiv P$ in $H^{m+1}[-\pi,\pi]$ (D is ordinary differentiation, $D - aI\colon H^{m+1}[-\pi,\pi] \to H^m[-\pi,\pi]$), and the range of $(A - aI)^{-1}$ is in $H^{m+1}[-\pi,\pi]$). Thus

$$\hat{x} = (A - aI)^{-1}(D - aI)x \equiv Px$$

$$\tilde{x}(t) = e^{at}((I - P)x)(0) \ , \quad \tilde{x} \equiv Q\,x \ .$$

It is easily seen that P and Q are projections, not in general orthogonal, such that

$$PQ = QP = 0 \ , \quad P + Q = I \ .$$

Correspondingly we have dual projections P^*, Q^* onto dual subapaces $\tilde{X}'$, $\hat{X}'$ in $H^{m+1}[-\pi,\pi]'$. The space $\tilde{X}'$ is one dimensional. Let η be the unique element of unit norm in $\tilde{X}'$. Then

$$\langle x,\eta \rangle = 0$$

for all $x \in \hat{X}$ while

$$\langle e^{a\cdot},\eta \rangle \neq 0 \ .$$

We now show that

$$\mathcal{D}(A) = \hat{X} = \{x \in H^{m+1}[-\pi,\pi] \mid \langle x,\eta \rangle = 0 \} \ ,$$

The second equality is clear. As for the first, it is enough to show that $\hat{X}$ = range of $(A - aI)^{-1}$. But this follows because, if, $x \in (A - aI)^{-1}$, $(D - aI)x = (A - aI)x$ and $Px = x$. That $\eta \notin H^m[-\pi,\pi]'$ follows from the fact that $R(A - aI')^{-1} = \mathcal{D}(A)$ is dense in $H^m[-\pi,\pi]$. The proof is now complete.

Theorem 3.7. Let $x \in H^m[-\pi,\pi]$ and let

$$\psi(\lambda) = L_A(x,\lambda)$$

be the L_A-transform of x. Then

$$\psi(\lambda) = \psi_0(\lambda) + \psi_1(\lambda)$$

where $\psi_0(\lambda)$ is a polynomial of degree at most m in $1/\lambda$ with no constant term:

$$\psi_0(\lambda) = \frac{\psi_1}{\lambda} + \frac{\psi_2}{\lambda^2} + \cdots + \frac{\psi_m}{\lambda^m}, \qquad (3.38)$$

and $\psi_1(\lambda)$ is a meromorphic function of λ, holomorphic outside $\sigma(A) \cup \{0\}$ and having the following properties:

(i) with $\lambda = \xi + i\eta$, there exists $M(\rho) > 0$ such that

$$\|(1 + |\xi + i\,\cdot|)^m \psi_1(\xi + i\cdot)\|_{L^2(-\infty,\infty)} \leq M(\rho)\|x\|_{H^m[-\pi,\pi]} \qquad (3.39)$$

$\xi > \rho > \gamma^+$, $\xi < -\rho < \gamma^-$.

(ii) Let $\Gamma = \Gamma_0 \cup \Gamma_1$ be a contour consisting of the small circle, Γ_0, centered at 0 and lying in $\rho(A)$ together with

$$\Gamma_1 = \{\xi + i\eta \mid -\infty < \eta < \infty,\ \xi = \pm\rho,\ \rho > \max(\gamma^+, -\gamma^-)\},$$

both oriented in the positive direction. Then

$$x_A(t) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\psi_0(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}\psi_1(\lambda)\,d\lambda, \qquad (3.40)$$

the first integral being void and the second taken in the l.i.m. sense if $m = 0$. Defining

$$\|\psi\|^2_{\rho,A} = \int_{\Gamma_0} |\psi_0(\lambda)|^2 |d\lambda| + \int_{\Gamma_1} |(1 + |\lambda|)^m (\psi_1(\lambda)|^2 |d\lambda|, \tag{3.41}$$

$L_A : H^m[-\pi,\pi] \to \Phi_A (\equiv L_A(H^m[-\pi,\pi])$ <u>is bounded and boundedly invertible.</u>

(iii) <u>Let</u> $y \in H^m[-\pi,\pi]'$ <u>with</u> F-<u>transform</u> $\phi(\lambda)$. <u>Then</u>

$$(x,y) = \frac{1}{2\pi i} \int_{\Gamma_0} \varphi(\lambda)\psi_0(\lambda)d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} \varphi(\lambda)\psi_1(\lambda)d\lambda) \tag{3.42}$$

<u>independent of the particular operator</u> A <u>used to define</u> $\psi(\lambda) = L_A(x,\lambda)$.

<u>Remark.</u> It is also true that

$$x_A(t) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda t}\psi(\lambda)d\lambda \tag{3.40a}$$

in the l.i.m. sense, but (3.40) is absolutely convergent and yields more information about the behavior of $x_A(t)$.

<u>Proof.</u> Since

$$\psi(\lambda) = \int_0^\infty e^{-\lambda t} x_A(t)dt, \qquad Re(\lambda) > \gamma^+ ,$$

and since $x_A(t)$ has the properties noted in Proposition 3.3, we may integrate by parts repeatedly to obtain

$$\psi(\lambda) = \frac{x(0)}{\lambda} + \frac{x'(0)}{\lambda} + \cdots + \frac{x^{(m-1)}(0)}{\lambda^m} + \frac{1}{\lambda^m} \int_0^\infty e^{-\lambda t} x_A^{(m)}(t)dt$$
$$\equiv \psi_0(\lambda) + \psi_1(\lambda) . \tag{3.43}$$

A similar computation is valid for $Re(\lambda) < \gamma^-$. Since $\psi_0(\lambda)$ is holomorphic except at 0 and $\psi(\lambda)$ is holomorphic outside $\sigma(A)$, $\psi_1(\lambda)$ is holomorphic outside $\sigma(A)$ $\{0\}$ as claimed. Also from Proposition 3.3 we see that

$$e^{-\gamma^+ t} x_A^{(m)}(t) \in L^2[0,\infty) ,$$

and then

$$(1 + |\lambda|^m \psi_1(\lambda) = \left(\frac{1 + |\lambda|^m}{\lambda^m} \right) \int_0^\infty e^{-(\lambda-\gamma^+)t} e^{-\gamma^+ t} x_A^{(m)}(t)dt \tag{3.44}$$

is, by virtue of Plancherel's theorem, square integrable on any vertical line in the complex plane lying to the right of $Re(\lambda) = \gamma^+$. A similar result holds for $Re(\lambda) < \gamma^-$. The inequality (3.39) follows from the boundedress of $(1 + |\lambda|^m/\lambda^m$ on Γ_1 , Plancherel's

theorem and (3.44).

Formula (3.40) is just a modification of the familiar Laplace inversion formula. For $\mathrm{Re}(\lambda) = \xi > \gamma^+$ we have

$$\underset{R\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi}\int_{-R}^{R} e^{(\xi+i\eta)t}\psi(\xi+i\eta)d\eta = \begin{cases} 0 , & t < 0 \\ x_A(t), & t > 0 \end{cases}$$

and for $\mathrm{Re}(\lambda) = \xi < \gamma^-$ we have

$$\underset{R\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi}\int_{R}^{+R} e^{(\xi+i\eta)t}\psi(\xi+i\eta)d\eta = \begin{cases} x_A(t) , & t < 0 \\ 0 , & t > 0 . \end{cases}$$

(The situation at $t = 0$ as actually slightly more complicated but those complications have no bearing on what we do here.) Thus if $\Gamma_{1,R}$ consists of the two line segments

$$\Gamma_{1,R} = \{\xi+i\eta) | \xi=\rho > \gamma^+ , \ |\eta| \leq R\}$$
$$\cup \{\xi+i\eta | \xi = -\rho < \gamma^- , \ |\eta| \leq R\}$$

we can write

$$x_A(t) = \underset{R\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi i}\int_{\Gamma_{1,R}} e^{\lambda t}\psi(\lambda)d\lambda =$$
$$= \underset{R\to\infty}{\mathrm{l.i.m.}}\ \left[\frac{1}{2\pi i}\int_{\Gamma_{\rho,R}} e^{\lambda t}\psi_0(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_{1,R}} e^{\lambda t}\psi_1(\lambda)d\lambda \right] . \tag{3.45}$$

Since $\psi_1(\lambda)$ has the form shown in (3.38), (3.43), the integral of $e^{\lambda t}\psi_0(\lambda)$ over paths joining $\rho + iR$ to $-\rho + iR$, $-\rho - iR$ to $\rho - iR$, tend to zero as $R \to \infty$ and (3.45) gives

$$x_A(t) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\psi_0(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}\psi_1(\lambda)d\lambda . \tag{3.46}$$

When $m = 0$ the first integral is void and the second must still be interpreted in the l.i.m. sense. Thus (3.40) is proved.

That the right hand side of (3.41) can be bounded by a positive multiple of $\|x\|_{H^m[-\pi,\pi]}$ follows from (3.19), (3.44) and the form (3.43) of $\psi_0(\lambda)$, together with the fact that $\sum_{k=0}^{m-1} |x^{(k)}(0)|^2 \leq \|x\|_{H^m[-\pi,\pi]}$. The estimate in the other direction is obtained by noting that Plancherel's theorem allows one to bound $\|x^{(m)}\|_{L^2[-\pi,\pi]}$ in terms of $(1 + |\lambda|^m)\psi_1(\lambda)$, and the Schwartz inequality applied to

$$x^{(k)}(0) = \frac{1}{2\pi i} \int_{\Gamma_0} \lambda^k \, \psi_0(\lambda) d\lambda$$

enables us to estimate $\sum_{k=0}^{m-1} |x^{(k)}(0)|^2$ in terms of the first integral in (3.41). We may regard (ii) as established.

Finally there is (3.42) to prove. Let

$$y = (y_0, y_1, \cdots, y_{m-1}, y_m(\cdot)) \in H^m[-\pi,\pi]'$$

and let $\varphi(\lambda)$ be its F-transform, as discussed earlier. For

$$x = (x(0), x'(0), \cdots, x^{(m-1)}(0), x^{(m)}(\cdot)) \in H^m[-\pi,\pi]$$

we have

$$\langle x,y \rangle = \sum_{k=0}^{m-1} y_k x^k(0) + \int_{-\pi}^{\pi} y_m(s) x^{(m)}(s) ds. \tag{3.47}$$

Letting ψ be the L_A-transform of x, the estimates (3.8), (3.39) allow us to form

$$\frac{1}{2\pi i} \int_{\Gamma_0} \varphi(\lambda)\psi_0(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} \varphi(\lambda)\psi_1(\lambda) d\lambda$$

$$= \frac{1}{2\pi i} \int_{\Gamma_0} \Big[\sum_{k=0}^{m-1} y_k \lambda^k + \lambda^m \int_{-\pi}^{\pi} e^{\lambda s} y_m(s) ds\Big]\psi_0(\lambda) d\lambda$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_1} \Big[\sum_{k=0}^{m-1} y_k \lambda^k + \lambda^m \int_{-\pi}^{\pi} e^{\lambda s} y_m(s) ds\Big]\psi_1(\lambda) d\lambda .$$

Differentiation of (3.40) with respect to t shows that

$$\frac{1}{2\pi i} \int_{\Gamma_0} \Big(\sum_{k=0}^{m-1} y_k \lambda^k\Big)\psi_0(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} \Big(\sum_{k=0}^{m-1} y_k \lambda^k\Big)\psi_1(\lambda) d\lambda$$

$$= \sum_{k=0}^{m-1} y_k \, x^{(k)}(0). \tag{3.48}$$

The term

$$\frac{1}{2\pi i}\int_{\Gamma_0}\left(\lambda^m\int_{-\pi}^{\pi} e^{\lambda s}\, y_m(s)\,ds\right)\psi_0(\lambda)\,d\lambda \equiv 0 \tag{3.49}$$

because $\lambda^m\psi_0(\lambda)$ and $\int_{-\pi}^{\pi} e^{\lambda s}\, y_m(s)\,ds$ are holomorphic inside Γ_0. Finally we note that $\int_{-\pi}^{\pi} e^{\lambda s}\, y_m(s)\,ds$ and $\lambda^m\psi_1(\lambda)$ are both square integrable on Γ_1, so the product is integrable and

$$\frac{1}{2\pi i}\int_{\Gamma_\rho}\lambda^m\int_{-\pi}^{\pi} e^{\lambda s} y_m(s)\,ds\;\psi_1(\lambda)\,d\lambda$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{\rho,R}}\lambda^m\int_{-\pi}^{\pi} e^{\lambda s} y_m(s)\,ds\;\;\psi_1(\lambda)\,d\lambda$$

$$= \lim_{R\to\infty}\int_{-\pi}^{\pi} y^m(s)\left(\frac{1}{2\pi i}\int_{\Gamma_{\rho,R}}\lambda^m e^{\lambda s}\psi_1(\lambda)\,d\lambda\right)ds$$

$$= \int_{-\pi}^{\pi} y_m(s)\left[\underset{R\to\infty}{\mathrm{l.i.m.}}\;\frac{1}{2\pi i}\int_{\Gamma_{\rho,R}}\lambda^m e^{\lambda s}\psi_1(\lambda)\,d\lambda\right]ds$$

$$= \int_{-\pi}^{\pi} y_m(s)\,x^{(m)}(s)\,ds, \tag{3.50}$$

the last two steps being valid because $\frac{1}{2\pi i}\int_{\Gamma_{\rho,R}}\lambda^m e^{\lambda a}\psi_1(\lambda)\,d\lambda$ converges in $L^2[-\pi,\pi]$ to $x^{(m)}(s)$ - using the Laplace inversion formula with

$$\lambda^m\psi_1(\lambda) = \int_0^\infty e^{-\lambda s}x_A^{(m)}(s)\,ds,\quad x_A^{(m)}(s) = x^{(m)}(s),\quad s\in[-\pi,\pi]\;.$$

Combining (3.48), (3.49), (3.50) we have (3.42). Since the left hand side of (3.42) is independent of A, the right hand side is also. With this, the proof of Theorem 3.6 is complete.

The fact that (3.42) is independent of A, more accurately, independent of $\mathcal{D}(A)$, enables us to define the L transform of $x \in H^m[-\pi,\pi]$ as a certain equivalence class of meromorphic functions.

Definition 3.8. For each $x \in H^m[-\pi,\pi]$, the L-transform of x, $L(x)$, consists of all functions $\psi(\lambda)$ (not necessarily obtained by means of (3.22), (3.23) for some A) which

are holomorphic on Γ_1 for some $\rho > 0$, meromorphic in the interior of Γ_1, and have a decomposition

$$\psi(\lambda) = \psi_0(\lambda) + \psi_1(\lambda)$$

with $\psi_0(\lambda)$ of the form (3.38), $\lambda^m \psi_1(\lambda)$ square integrable on Γ_ρ and such that, with

$$\hat{\psi}(\lambda) = \hat{\psi}_0(\lambda) + \hat{\psi}_1(\lambda) = L_{\hat{A}}(x,\lambda)$$

for $\hat{A}$ an arbitrary neutral generator on $H^m[-\pi,\pi]$,

$$\frac{1}{2\pi i}\int_{\Gamma_0} \varphi(\lambda)[\psi_0(\lambda) - \hat{\psi}_0(\lambda)]\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} \varphi(\lambda)[\psi_1(\lambda) - \hat{\psi}_1(\lambda)]\,d\lambda = 0$$

for all $\varphi \in F(H^m[-\pi,\pi]')$.

It is clear that $L(x)$ includes every function $L_A(x,\lambda)$, A neutral generator on $H^m[-\pi,\pi]$.

We can define a norm on $L(H^m[-\pi,\pi])$ by

$$\|L(x)\|_\rho = \sup_{\|\varphi\|_\rho = 1} \left| \frac{1}{2\pi i}\int_{\Gamma_0} \varphi(\lambda)\psi_0(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} \varphi(\lambda)\psi_1(\lambda)\,d\lambda \right|$$

and it may be seen that with this norm $L: H^m[-\pi,\pi] \to L(H^m[-\pi,\pi])$ is bounded and boundedly invertible.

It should be noted that (3.42) is not true for all holomorphic $\varphi(\lambda)$ for which (3.42) is convergent, but only for $\varphi \in F(H^m[-\pi,\pi]')$. Consider the following example. Take $m = 1$, let A be differentiation (2.9) on $H^1[-\pi,\pi]$ with domain

$$D(A) = \{x \in H^2[-\pi,\pi] \,|\, x'(\pi) = x'(-\pi)\}\ .$$

Let $\tilde{A}$ be the same operator but with domain

$$D(\tilde{A}) = \{x \in H^2[-\pi,\pi] \,|\, x'(\pi) = -x'(-\pi)\}\ .$$

Let

$$\varphi(\lambda) = \int_\pi^{3\pi} \lambda e^{\lambda t}\,dt\ .$$

Then, with $\psi(\lambda) = \psi_0(\lambda) + \psi_1(\lambda)$ being the L_A transform of $x \in H^1[-\pi,\pi]$, with $x(\pi) = x(-\pi) = 0$, we compute

$$\frac{1}{2\pi i}\int_{\Gamma_0} \varphi(\lambda)\psi_0(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} \varphi(\lambda)\psi_1(\lambda)d\lambda = \int_{\pi}^{3\pi} x_A'(t)dt = \int_{-\pi}^{\pi} x'(t)dt$$

while, letting $\tilde{\psi}(\lambda) = \tilde{\psi}_0(\lambda) + \tilde{\psi}_1(\lambda)$ be $L_{\tilde{A}}(x,\lambda)$,

$$\frac{1}{2\pi i}\int_{\Gamma_0} \varphi(\lambda)\tilde{\psi}_0(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} \varphi(\lambda)\tilde{\psi}_1(\lambda)d\lambda = \int_{\pi}^{3\pi} x_{\tilde{A}}'(t)dt = \int_{-\pi}^{\pi} (-x'(t))dt .$$

The behavior of the integral (3.42) for φ which are F transforms of distributions with supports extending beyond $[-\pi,\pi]$ will occupy us extensively in Section 4.

One of the reasons why $L(x)$ is defined so broadly in Definition 3.8 is so that we may assert that

$$L(H^{m+1}[-\pi,\pi] \subseteq L(H^m[-\pi,\pi]). \tag{3.51}$$

If $\psi \in L(H^{m+1}[-\pi,\pi])$,

$$\begin{aligned}\psi(\lambda) &= \frac{\psi_0}{\lambda} + \frac{\psi_1}{\lambda^2} + \cdots = \frac{\psi_{m-1}}{\lambda^m} + \frac{\psi_m}{\lambda^{m+1}} + \frac{1}{\lambda^{m+1}}\psi_1(\lambda) \\ &= \frac{\psi_0}{\lambda} + \frac{\psi_1}{\lambda^2} + \cdots + \frac{\psi_{m-1}}{\lambda^m} + \frac{1}{\lambda^m}\left[\frac{\psi_m}{\lambda} + \frac{1}{\lambda}\psi_1(\lambda)\right] \\ &\equiv \tilde{\psi}(\lambda) \in L(H^m[-\pi,\pi])\end{aligned}$$

according to our definition. To work with $L_A(x,\lambda)$ would be difficult because if A is a neutral generator on $H^{m+1}[-\pi,\pi]$ and if, for $x \in H^{m+1}[-\pi,\pi]$, we extend x to x_A and obtain $L_A(x,\lambda)$, there is not, in general, any $\tilde{A}$ a neutral generator on $H^m[-\pi,\pi]$ for which $x_{\tilde{A}} = x_A$ and we cannot, therefore, say $L_A(x,\lambda) = L_{\tilde{A}}(x,\lambda) \in L_{\tilde{A}}(H^m[-\pi,\pi])$. If (3.51) is to be true, $L(H^m[-\pi,\pi])$ must consist of equivalence classes of functions more general than those of the form $L_A(x,\lambda)$.

Let A be a neutral generator on $H^m[-\pi,\pi]$ and let η be the associated generating functional in $H^{m+1}[-\pi,\pi]'$, whose existence has been established in Theorem 3.6. Let $x \in H^m[-\pi,\pi]$ and let $\psi(\lambda) = L_A(x,\lambda)$. The poles of $\psi(\lambda)$ occur only at values λ_j

which are eigenvalues of the operator A. If λ_j is simple, the associated eigenfunction of A must be $e^{\lambda_j s}$. If λ_j has multiplicity m_j, then $e^{\lambda_j s}$, $se^{\lambda_j s}$, $(s^2/2)e^{\lambda_j s}, \cdots, (s^{m_j-1}/(m_j-1)!)e^{\lambda_j s}$ are associated generalized eigenfunctions and

$$A\left(\sum_{k=1}^{m_j} \alpha_k \frac{s^{k-1}}{(k-1)!} e^{\lambda_j s}\right) = \sum_{k=1}^{m_j} \beta_k \frac{s^{k-1}}{(k-1)!} e^{\lambda_j s}$$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{m_j} \end{pmatrix} = A_j \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m_j} \end{pmatrix}, \quad A_j = \begin{pmatrix} \lambda_j & 1 & \cdots & 0 & 0 \\ 0 & \lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & \cdots & 0 & \lambda_j \end{pmatrix},$$

because A agrees with differentiation on its domain. These properties are all reflected in properties of the generating functional η and its F-transform, $\varphi(\lambda) = F(\eta,\lambda)$. A simple eigenvalue, λ_i, for A corresponds to $\varphi(\lambda) = \langle e^{\lambda s},\eta\rangle$ having a simple zero at λ_j, while an eigenvalue λ_j of multiplicity m_j is such that $\varphi(\lambda)$ has a zero of multiplicity m_j at λ_j. The equations

$$\left\langle \frac{s^{k-1}}{(k-1)!} e^{\lambda_j s}, \eta \right\rangle = 0 \quad k = 1,2,\cdots,m ,$$

are the same as

$$\frac{1}{(k-1)!} \frac{d^{k-1}}{(ds)^{k-1}} \langle e^{\lambda s},\eta \rangle \Big|_{\lambda=\lambda_j} = \frac{1}{(k-1)!} \varphi^{(k-1)}(\lambda_j) = 0 , \quad k = 1,2,\cdots,m_j .$$

We show now, at least for a certain class of generating functionals λ, that the poles, with their associated multiplicities and residues, of $\psi(\lambda) = L_A(x,\lambda)$ correspond to expansions of x in terms of exponential, or generalized exponential bases for $H^m[-\pi,\pi]$.

<u>Theorem 3.9.</u> <u>Assume that the region,</u> R <u>enclosed by the contour</u> Γ_1 <u>of</u> Theorem 3.7 <u>can be divided into subregions</u> R_j, $-\infty < j < \infty$, <u>by means of paths</u> c_j <u>joining the line</u> $\mathrm{Re}(\lambda) = \rho$ <u>to the line</u> $\mathrm{Re}(\lambda) = -\rho$, <u>oriented in that direction, so that</u> c_j, <u>the portion of</u> $\mathrm{Re}(\lambda) = -\rho$ <u>between</u> c_j <u>and</u> c_{j-1}, $-c_{j-1}$ <u>and the portion of</u> $\mathrm{Re}(\lambda) = \rho$ <u>between</u> c_{j-1} <u>and</u> c_j <u>form a closed contour</u> C_j <u>surrounding finitely many eigenvalues</u>

$\lambda_{j,1}, \lambda_{j,2}, \cdots, \lambda_{j,n_j}$ of the operator A, the C_j having disjoint interiors. Assume further that for each $x \in H^m[-\pi,\pi]$, with $\psi(\lambda) = L_A(x,\lambda)$

$$\lim_{|j|\to\infty} \int_{C_j} |\lambda^m \psi_1(\lambda)||d\lambda| = 0 . \tag{3.52}$$

Then, letting $m_{j,\ell}$ be the multiplicity of $\lambda_{j,\ell}$, $\ell = 1,2,\cdots,n_j$, we have the unique representation, convergent in $H^m[-\pi,\pi]$,

$$x = \sum_{j=-\infty}^{\infty} x_j \tag{3.53}$$

$$x_j(s) = \sum_{\ell=1}^{n_j} \Big(\sum_{k=1}^{m_{j,\ell}} \psi_{-k}(\lambda_{j,\ell}) \frac{s^{k-1}}{(k-1)!} e^{\lambda_{j,\ell} s} \Big) ,$$

where $\psi_{-k}(\lambda_{j,\ell})$ is the coefficient of λ^{-k} in the Laurent expansion of $\psi(\lambda)$ about the pole $\lambda_{j,\ell}$. Thus the spaces X_j spanned by the functions $\frac{s^{k-1}}{(k-1)!} e^{\lambda_{j,\ell} s}$, $\ell = 1,2,\cdots,n_j$, $k = 1,2,\cdots,m_{j,\ell}$, form a strong decomposition of $H^m[-\pi,\pi]$.

Proof. Very little is required here except the inversion formula (3.40), which for $s \in [-\pi,\pi]$ reads

$$x(s) = \frac{1}{2\pi i} \int_{\Gamma_0} e^{\lambda s} \psi_0(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda s} \psi_1(\lambda) d\lambda ;$$

the significant work lies in verifying (3.52), which leads us to the work of Sections 4,5.

We may assume the *circle* Γ_0 *chosen so small that it excludes any non-zero* eigenvalues of A. If $\lambda_0 \equiv 0$ is an eigenvalue of A of multiplicity m_0, the first integral becomes

$$\sum_{k=1}^{m_0} \psi_{0,-k}(0) \frac{s^{k-1}}{(k-1)!} . \tag{3.54}$$

We write $\Gamma_1 = \hat{\Gamma}_{1,j} + \tilde{\Gamma}_{1,j}$ for each j, where $\hat{\Gamma}_{1,j} = \sum_{|i|<j} C_j$. Then

$$\frac{1}{2\pi i} \int_{\Gamma_1} e^{\lambda s} \psi_1(\lambda) d\lambda = \frac{1}{2\pi i} \int_{\hat{\Gamma}_{1,j}} e^{\lambda s} \psi_1(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\tilde{\Gamma}_{1,j}} e^{\lambda s} \psi_1(\lambda) d\lambda .$$

The first integral, since it encloses finitely many poles of $\psi_1(\lambda)$, can be written as

$$\sum_{i=-j}^{i=j} \Big(\sum_{\ell=1}^{n_j} \Big(\sum_{k=1}^{m_{j,\ell}} \psi_{1,-k}(\lambda_{j,\ell}) \frac{s^{k-1}}{(k-1)!} e^{\lambda_{j,\ell} s} \Big)\Big) \tag{3.55}$$

where $\psi_{1,-k}(\lambda_{j,\ell})$ is the coefficient of $\dfrac{1}{(\lambda - \lambda_{j,\ell})^k}$ in the Laurent expansion of ψ_1

about the pole $\lambda_{j,\ell}$. Since $\psi_0(\lambda)$ is holomorphic except at 0,

$$\psi_{1,-k}(\lambda_{j,\ell}) = \psi_{-k}(\lambda_{j,\ell})$$

except for the possible case of a zero eigenvalue. For that case the corresponding term $\psi_{-k}(0) = \psi_{0,-k}(0) + \psi_{1,-k}(0)$, the first term coming from (3.54), the second included in (3.55). Thus

$$\frac{1}{2\pi i} \int_{\Gamma_0} e^{\lambda s} \psi_0(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\hat{\Gamma}_{1,j}} e^{\lambda s} \psi_1(\lambda) d\lambda$$

$$= \sum_{i=-j}^{i=j} \Big(\sum_{\ell=1}^{n_j} \Big(\sum_{k=1}^{m_{j,\ell}} \psi_{-k}(\lambda_{j,\ell}) \frac{s^{k-1}}{(k-1)!} e^{\lambda_{j,\ell} s} \Big)\Big) .$$

To obtain (3.53) it is only necessary to show that, with

$$\tilde{x}_j(s) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_{1,j}} e^{\lambda z} \psi_1(\lambda) d\lambda$$

we have

$$\lim_{j \to \infty} \|\tilde{x}_j\|_{H^m[-\pi,\pi]} = 0 . \tag{3.56}$$

Differentiation gives

$$\tilde{x}_j^{(k)}(s) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_{1,j}} e^{\lambda s} \lambda^k \psi_1(\lambda) d\lambda, \quad k = 0,1,\cdots,m,$$

the validity of differentiation under the integral sign being assured by (3.39) of Theorem 3.7. Since $e^{\lambda s}$ is uniformly bounded for $s \in [-\pi,\pi]$ when $\mathrm{Re}(\lambda)$ is bounded, the fact that

$$\lim_{j \to \infty} \frac{1}{2\pi i} \int_{C_j(C_{-j})} e^{\lambda s} \lambda^k \psi_1(\lambda) d\lambda = 0 , \quad k = 0,1,2,\cdots,m, \tag{3.57}$$

uniformly for $s \in [-\pi,\pi]$, follows from (3.52).

Let us define $\Psi_j(\lambda)$, $\lambda \in \Gamma_1$, by

$$\Psi_j(\lambda) = \begin{cases} \Psi_1(\lambda), & \lambda \in \Gamma_1 \cap \tilde{\Gamma}_{1,j} \\ 0 \quad , & \lambda \in \Gamma_1 - \tilde{\Gamma}_{1,j} \end{cases} .$$

Then $\tilde{x}_j^{(k)}(s)$ is equal to the sum of the integrals in (3.57) plus the sum

$$\frac{1}{2\pi i} \int_{\Gamma_1^+} e^{\lambda s} \lambda^k \Psi_j(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1^-} e^{\lambda s} \lambda^k \Psi_j(\lambda) d\lambda , \tag{3.58}$$

$\Gamma_1^+ = \{\lambda | \mathrm{Re}(\lambda) = \rho\}$, $\Gamma_1^- = \{\lambda | \mathrm{Re}(\lambda) = -\rho\}$. Writing $\lambda = \xi + i\eta$, the first integral is the k-th derivative of the inverse Fourier transform of $\Psi_j(\rho + i\eta)$, $-\infty < \eta < \infty$, multiplied by $e^{\rho s}$. Since $\lambda^k \Psi_1(\lambda) \in L^2_{\Gamma_1^+}$, $k = 0,1,2,\cdots.m$, the Plancherel theorem shows that, as a function of s, the first integral in (3.58) is in $H^m(-\infty,\infty)$. Since the c_j tend to infinity as $|j| \to \infty$, it is clear that

$$\lim_{j\to\infty} \| e^{i\cdot s}(\rho + i\cdot)^k \Psi_j(\rho + i\cdot)\|_{L^2_{\Gamma_1^+}} = 0 , \quad k = 0,1,\cdots,m,$$

and, applying the Plancherel theorem again

$$\lim_{j\to\infty} \| \int_{\Gamma_1^+} e^{\lambda\cdot} \lambda^k \Psi_j(\lambda) d\lambda \|_{L^2[-\pi,\pi]} \leq \lim_{j\to\infty} \| \int_{\Gamma_1^+} e^{\lambda\cdot} \lambda^k \Psi_1(\lambda) d\lambda \|_{L^2(-\infty,\infty)} = 0, \tag{3.59}$$

$$k = 0,1,2,\cdots,m .$$

A similar result is obtained for the second integral in (3.58) and, combined with (3.57) and (3.59) we finally have (3.53), completing the proof of the theorem.

4. Functional Calculus, Convolution, Connections with Control and Observation Theory.

Let A be a neutral generator on $H^m[-\pi,\pi]$. Let $x \in H^m[-\pi,\pi]$ and let $\psi(\lambda) = L_A(x,\lambda)$. If $S(t)$ is the group generated by A, then $S(t)\,x \in H^m[-\pi,\pi]$ for any real t. There occurs to us then the question: What is the relationship between $L_A(x,\lambda)$ and $L_A(S(t)x,\lambda)$? Since $S(t) = e^{At}$, our first thought might be that $L_A(S(t)x,\lambda)$ should equal $e^{\lambda t}\psi(\lambda)$, but it turns out that this is not the case - indeed $e^{\lambda t}\psi(\lambda)$ is not in general the Laplace transform of any element in $H^m[-\pi,\pi]$.

In the preceding section we have developed the inversion formula (3.40) for $L_A(x,\lambda)$. But it is clear that if $\vartheta(\lambda)$ is any function meromorphic in the interior of

$$\Gamma_1 = \{\lambda \mid \mathrm{Re}(\lambda) = \rho\} \cup \{\lambda \mid \mathrm{Re}(\lambda) = -\rho\}$$

and analytic on Γ_1 itself, and if

$$\vartheta(\lambda) = \vartheta_0(\lambda) + \vartheta_1(\lambda)\ , \tag{4.1}$$

$\vartheta_0(\lambda)$ having a pole of order $\leq m$ at the origin, otherwise holomorphic on $\Gamma_1 \cup \mathrm{Int}(\Gamma_1)$ $(\supseteq \Gamma_0)$ and $\vartheta_1(\lambda) = \frac{1}{\lambda^m}\,\tilde{\vartheta}_1(\lambda)$, $\tilde{\vartheta}_1(\lambda) \in L^2(\Gamma_1)$, we can form

$$\xi(t) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\vartheta_0(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}\,\vartheta_1(\lambda)\,d\lambda\ . \tag{4.2}$$

The function $\xi(t)$ will have exponential growth in the right and left half planes and, for $\mathrm{Re}(\mu) > \rho$ or for $\mathrm{Re}(\mu) < -\rho$, we can form the Laplace transform of $\xi(t)$:

$$\begin{aligned}
\theta(\mu) &= \int_0^\infty e^{-\mu t}\,\xi(t)\,dt \\
&= \int_0^\infty e^{-\mu t}\left[\frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda t}\,\vartheta_0(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}\,\vartheta_1(\lambda)\,d\lambda\right]dt \\
&= \frac{1}{2\pi i}\int_{\Gamma_0}\frac{\vartheta_0(\lambda)}{\mu-\lambda}\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\vartheta_1(\lambda)}{\mu-\lambda}\,d\lambda \\
&\equiv \theta_0(\mu) + \theta_1(\mu)\ .
\end{aligned} \tag{4.3}$$

If we let

$$\tilde{\xi}(t) = \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda t}\,\vartheta_1(\lambda)\,d\lambda$$

then the fact that $\lambda^m \vartheta_1(\lambda) \in L^2(\Gamma_1)$ shows that $e^{-\rho t}\tilde{\xi}(t) \in H^m[0,\infty]$, $e^{\rho t}\tilde{\xi}(t)$ $\in H^m(-\infty,0]$. Since the first integral in (4.2) is just a polynomial in t, this remains true with $\tilde{\xi}(t)$ replaced by $\xi(t)$. Integrating by parts, as in (3.43), we find that

$$\theta(\mu) = \frac{\xi(0)}{\mu} + \cdots + \frac{\xi^{(m-1)}(0)}{\mu^m} + \frac{1}{\mu^m}\ \tilde{\theta}_1(\mu)$$

$$\equiv \theta_0(\mu) + \theta(\mu)\ .$$

When $\vartheta(\lambda)$ is the Laplace transform of a function ξ, the formula (4.2) returns the function ξ and $\theta = \vartheta$. Thus the map

$$P : \vartheta \to \theta \tag{4.4}$$

is a projection, easily seen to be bounded with respect to $\|\vartheta\|_\rho \equiv \left[\|\vartheta_0\|^2_{L^2(\Gamma_0)} + \|(1 + |\lambda|)^m\ \vartheta_1(\lambda)\|^2_{L^2(\Gamma_1)}\right]^{\frac{1}{2}}$ on the space of meromorphic functions (4.1), the range of P consisting of Laplace transforms which have that form.

The characterization of $L_A(S(t)x,\lambda)$ is developed along these lines. Clearly, for x *(and hence $S(t)x$) in $H^m[-\pi,\pi]$*,

$$(S(t)x)_A(\tau) = (S(t + \tau)x)(0)\ .$$

Hence, for $\mathrm{Re}(\mu) > \rho$

$$L_A(S(t)x,\mu) = \int_0^\infty e^{-\mu\tau}(S(t + \tau)x)(0)d\tau\ .$$

But

$$(S(t + \tau)x)(0) = x_A(t + \tau) = \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda(t + \tau)}\psi_0(\lambda)d\lambda$$

$$+ \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda(t + \tau)}\psi_1(\lambda)d\lambda\ , \quad \psi(\lambda) = L_A(x,\lambda)\ .$$

Assuming $\mathrm{Re}(\mu) > \rho$ still,

$$L_A(S(t)x,\mu) = \int_0^\infty e^{-\mu\tau}\ \frac{1}{2\pi i}\int_{\Gamma_0} e^{\lambda(t + \tau)}\psi_0(\lambda)d\lambda d\tau$$

$$+ \int_0^\infty e^{\mu\tau}\ \frac{1}{2\pi i}\int_{\Gamma_1} e^{\lambda(t + \tau)}\psi_1(\lambda)d\lambda d\tau$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0} \frac{e^{\lambda t}\psi_0(\lambda)}{\mu - \lambda}\ d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{e^{\lambda t}\psi_1(\lambda)}{\mu - \lambda}\ d\lambda\ . \tag{4.5}$$

A similar calculation with the same result, is valid for $\mathrm{Re}(\mu) < -\rho$. It is easy to check that we also have

$$L_A(S(t)x,\mu) = (S(t)R(\mu,A)x)(0) = (e^{At}R(\mu,A)x)(0) \qquad (4.6)$$

and this provides the analytic continuation of $L_A(S(t)x,\mu)$ to $\mathbf{C} - \sigma(A)$. A further characterization of $L_A(S(t)x,\mu) = L_A(e^{At}x,\mu)$ is clear from (4.5):

$$L_A(e^{At}x,\mu) = P(e^{\mu t}\psi(\mu))$$

where P is the projection (4.4).

The success of this project suggests a slightly more ambitious one. Let $B(n)$ consist of functions f entire in the plane, satisfying in each strip $-R < \mathrm{Re}(\lambda) < R$ a polynomial growth condition, with M independent of R,

$$|f(\lambda)| \leq M(1 + |\lambda|)^n \qquad (4.7)$$

and let $B_1(n)$ be the subset of $B(n)$ for which the stronger condition

$$M_1(1 + |\lambda|)^n \leq |f(\lambda)| \leq M(1 + |\lambda|)^n \qquad (4.8)$$

is satisfied.

Our objective is the development of a convolution theory for the spaces $H^m[-\pi,\pi]$, $H^n[-\pi,\pi]'$ and its expression in terms of the transforms L_A and F. It is a curious and significant fact that the most important results (for our purposes) have to do with the case of an element $y \in H^{m+1}[-\pi,\pi]'$, with an additional restriction to be noted shortly, and an element $x \in H^m[-\pi,\pi]$.

Throughout the remainder of the section we will make use of the following system of contours, Γ_0, Γ_1, C_0, C_1. We assume that ρ is a positive number such that (cf.

$$-\rho < \gamma^- < \gamma^+ < \rho$$

and Γ_1 is the contour consisting of $\mathrm{Re}(\lambda) = \rho$, $\mathrm{Re}(\lambda) = -\rho$, positively oriented. The contour Γ_0 is a circle, centered at 0, lying in $\mathrm{Int}(\Gamma_1)$. The contours C_0, C_1 are similar, with ρ replaced by $\rho - \delta$, $\delta > 0$, and C_0 is inside Γ_0. A typical configuration is shown in Figure 4.1.

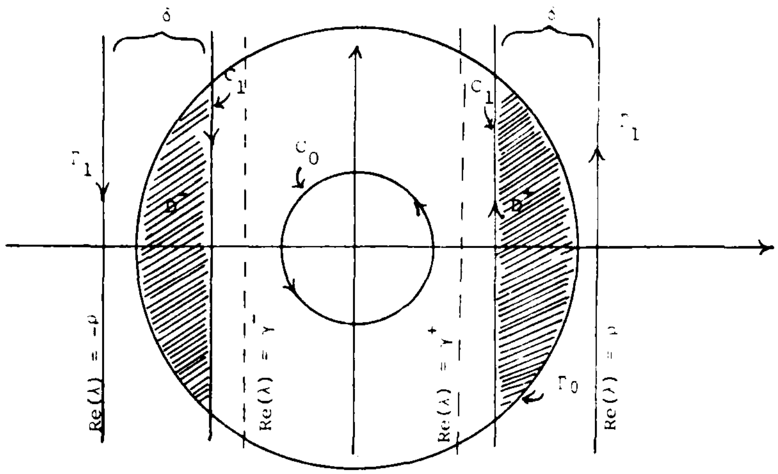

Figure 4.1

We shall suppose that $\varphi(\lambda)$ is the F-transform of an element $y \in H^{n+1}[-\pi,\pi]'$ with the additional property that $\varphi \in B(n)$ for $0 \le n \le m$ and we shall suppose that $\psi(\lambda) = L_A(x,\lambda)$ is the A-Laplace transform of an element $x \in H^m[-\pi,\pi]$, A being a neutral generator on $H^m[-\pi,\pi]$.

Lemma 4.1. With φ, ψ, Γ_0, Γ_1 as described above we have

$$\varphi(\lambda)\psi(\lambda) = \left[\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\varphi(\mu)\psi_0(\mu)}{\mu-\lambda}\,d\mu + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\varphi(\mu)\psi_1(\mu)}{\mu-\lambda}\,d\mu\right]$$

$$+\left[\frac{1}{2\pi i}\int_{C_0}\frac{\varphi(\mu)\psi_0(\mu)}{\lambda-\mu}\,d\mu + \frac{1}{2\pi i}\int_{C_1}\frac{\varphi(\mu)\psi_1(\mu)}{\lambda-\mu}\,d\mu\right]$$

$$\equiv Q(\varphi\psi)(\lambda) + P(\varphi\psi)(\lambda) \tag{4.9}$$

for λ in the shaded region $D = D^+ \cup D^-$ shown in Figure 4.1. Here as previously,

$$\psi_0(\lambda) = \frac{x(0)}{\lambda} + \cdots + \frac{x^{(m-1)}(0)}{\lambda^m},$$

$$\psi_1(\lambda) = \frac{1}{\lambda^m}\int_0^\infty e^{-\lambda t}\, x_A^{(m)}(t)\,dt .$$

Proof If Γ_D is the (positively oriented) boundary of D then it is clear that for $\lambda \in D$

$$\varphi(\lambda)\psi(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_D} \frac{\varphi(\mu)\psi(\mu)}{\mu - \lambda}\, d\mu$$

$$= \frac{1}{2\pi i}\int_{\Gamma_D} \frac{\varphi(\mu)\psi_0(\mu)}{\mu - \lambda}\, d\mu + \frac{1}{2\pi i}\int_{\Gamma_D} \frac{\varphi(\mu)\psi_1(\mu)}{\mu - \lambda}\, d\mu \,. \tag{4.10}$$

Because the only singularity of $\psi_0(\lambda)$ is at $\lambda = 0$ and $\varphi(\lambda)$ is entire

$$\frac{1}{2\pi i}\int_{\Gamma_D} \frac{\varphi(\mu)\psi_0(\mu)}{\mu - \lambda}\, d\mu = \frac{1}{2\pi i}\int_{\Gamma_0} \frac{\varphi(\mu)\psi_0(\mu)}{\mu - \lambda}\, d\mu + \frac{1}{2\pi i}\int_{C_0} \frac{\varphi(\mu)\psi_0(\mu)}{\lambda - \mu}\, d\mu \,. \tag{4.11}$$

Since the only singularities of $\psi_1(\mu)$ lie in the interior of C_1, it is clear that

$$\frac{1}{2\pi i}\int_{\Gamma_D} \frac{\varphi(\mu)\psi_1(\mu)}{\mu - \lambda}\, d\mu = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{\varphi(\mu)\psi_1(\mu)}{\mu - \lambda}\, d\mu \tag{4.12}$$

where Γ_N is the positively oriented boundary of the region which lies between Γ_1 and C_1 and the line $\mathrm{Im}(\lambda) = N$, $\mathrm{Im}(\lambda) = -N$. Since $|\varphi(\lambda)| \leq M(1 + |\lambda|)^n$ throughout the region under consideration and a variant of the Riemann-Lebesgue theorem shows that

$$\lim_{\substack{|\mathrm{Im}(\lambda)| \to \infty \\ |\mathrm{Re}(\lambda)| \leq \rho}} |\lambda|^m |\psi_1(\lambda)| = 0$$

and $m > n$ we have

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\Gamma_N} \frac{\varphi(\mu)\psi_1(\mu)}{\mu - \lambda}\, d\mu$$

$$= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\varphi(\mu)\psi_1(\mu)}{\mu - \lambda}\, d\mu + \frac{1}{2\pi i}\int_{C_1} \frac{\varphi(\mu)\psi_1(\mu)}{\lambda - \mu}\, d\mu \tag{4.13}$$

provided the latter two integrals are convergent - which is clear since $\varphi(\mu)\psi_1(\mu)$ and $\frac{1}{\lambda - \mu}$ are both square integrable on C_1 and Γ_1. Combining (4.10) - (4.13) we have (4.9) and the proof is complete.

Let us observe that, since $\varphi(\lambda)$ satisfies condition (4.7) in the strip $-R <$ $\mathrm{Re}\,(\lambda) < R$ for every $R > 0$, we may allow both ρ and $r_0 \equiv$ radius of Γ_0 to tend to infinity, keeping C_0 and C_1 fixed. As we do so, the formula for $P\,(\varphi\psi)(\lambda)$ is unchanged. Thus we may define

$$P(\varphi\psi)(\lambda) = \frac{1}{2\pi i} \int_{C_0} \frac{\varphi(\mu)\psi_0(\mu)}{\lambda - \mu} d\mu + \frac{1}{2\pi i} \frac{\varphi(\mu)\psi_1(\mu)}{\lambda - \mu} d\mu \tag{4.14}$$

for any λ lying to the right of the right hand half, C_1^+, of C_1 and for any λ lying to the left ofthe left hand half, C_1^-, of C_1. By varying C_1 it may be seen that

$$g(\lambda) \equiv P(\varphi(\psi)(\lambda)$$

may be defined this way for $Re(\lambda) > \gamma^+$ and for $Re(\lambda) < \gamma^-$. A meromorphic continuation of $g(\lambda)$ across and into the interior of C_1 may be obtained by noting that $\varphi(\lambda)\psi(\lambda)$ is meromorphic and (letting $r_{0,\rho}$ tend to infinity)

$$f(\lambda) \equiv Q(\varphi\psi)(\lambda)$$

is an entire function. Thus the identity

$$g(\lambda) = \varphi(\lambda)\psi(\lambda) - Q(\varphi\psi)(\lambda) = \varphi(\lambda)\psi(\lambda) - f(\lambda) \tag{4.15}$$

which (4.9) shows to be valid outside C_1 (taking $r_{0,\rho}$ sufficiently large) provides the desired meromorphic continuation. We see than that (4.9) extends to the whole plane, with the exception of the poles of $\psi(\lambda)$, enabling us to write $\varphi(\lambda)\psi(\lambda)$ as the sum of an entire function $f(\lambda) = Q(\varphi\psi)(\lambda)$ and a meromorphic function $g(\lambda) = P(\varphi\psi)(\lambda)$, This is obviously the same process as occurs in the development of Laurent series.

Proposition 4.2. There exists a function z $(= z_A)$, defined for $-\infty < t < \infty$, with

$$e^{-\lambda t} z(t) \in \begin{cases} H^{m-n}[0,\infty), & Re(\lambda) > \gamma^+ \\ H^{m-n}(-\infty,0], & Re(\lambda) < \gamma^- \end{cases} \tag{4.16}$$

such that $g(\lambda)$ is the Laplace transform of z. Further, there exists an element u $(=u_A) \in H^n[-\pi,\pi]'$ such that $f(\lambda)$ is the F-transform of u. Both z and u are uniquely determined by φ (hence y) and ψ (hence x and A).

Proof. Let $\Gamma_0, \Gamma_1, C_0, C_1$ be defined as in Lemma 4.1. If $Re(\lambda) > \gamma^+$ it may be arranged (possibly by making Γ_1 slightly asymmetric if $\gamma^+ \neq \gamma^-$) that the path Γ_1^+ lies to the right of C_1^+ and to the left of λ. Then

$$(Pg)(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{g(\mu)}{\lambda - \mu} \, d\mu$$

$$= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\lambda - \mu} \Big[\frac{1}{2\pi i} \int_{C_0} \frac{\varphi(\nu)\psi_0(\nu)}{\mu - \nu} d\nu + \frac{1}{2\pi i} \int_{C_1} \frac{\varphi(\nu)\psi_1(\nu)}{\mu - \nu} d\nu \Big]$$

$$= \frac{1}{2\pi i} \int_{C_0} \varphi(\nu)\psi_0(\nu) \Big(\frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\lambda - \mu} \frac{1}{\mu - \nu} d\mu \Big) d\nu$$

$$+ \frac{1}{2\pi i} \int_{C_1} \varphi(\nu)\psi_1(\nu) \Big(\frac{1}{2\pi i} \int_{\Gamma_1} \frac{1}{\lambda - \mu} \frac{1}{\mu - \nu} d\mu \Big) d\nu$$

$$= \frac{1}{2\pi i} \int_{C_0} \frac{\varphi(\nu)\psi_0(\nu)}{\lambda - \nu} d\nu = \frac{1}{2\pi i} \int_{C_1} \frac{\varphi(\nu)\psi_1(\nu)}{\lambda - \nu} d\nu$$

$$= g(\lambda) \; .$$

It follows that if we set

$$z(t) = \frac{1}{2\pi i} \int_{C_0} e^{\nu t} \varphi(\nu)\psi_0(\nu) d\nu + \frac{1}{2\pi i} \int_{C_1} e^{\nu t} \varphi(\nu)\psi_1(\nu) d\nu$$

that $g(\lambda)$ is the Laplace transform of $z(t)$. Since $\nu^{m-n}\varphi(\nu)\psi_1(\nu)$ is square integrable on C_1, for λ to the right of C_1 we have

$$e^{\lambda t} z^{(k)}(t) = \frac{1}{2\pi i} \int_{C_0} \nu^k e^{(\nu-\lambda)t} \varphi(\nu)\psi_0(\nu) d\nu) + \frac{1}{2\pi i} \int_{C_1} \nu^k e^{(\nu-\lambda)t} \varphi(\nu)\psi_1(\nu) d\nu \; , \qquad (4.17)$$

$$k = 0, \cdots, m-n \; ,$$

the last integral being taken in the l.i.m. sense if $m = n$. The first integral is $e^{\lambda t} \times$ a polynomial in t. The second is in $L^2[0,\infty)$ by the Plancherel theorem. We conclude that the left hand side of (4.17) is in $L^2[0,\infty)$ for $k = 0, \cdots, m-n$ and (4.16) follows.

To complete the proof we must establish that $Q(\varphi\psi)(\lambda)$ is the F-transform of an element $u \in H^n[-\pi,\pi]'$.

Using the fact that $\varphi(\lambda)$ satisfies (4.7) together with the fact that

$$\psi(\lambda) = \frac{x(0)}{\lambda} + \cdots + \frac{x^{(m-1)}(0)}{\lambda^m} + \frac{1}{\lambda^m} \int_0^\infty e^{-\lambda t} x^{(m)}(t) dt$$

is of order $\frac{1}{|\lambda|}$ we see that $\varphi(\lambda)\psi(\lambda)$ may be bounded by a constant times $(1 + |\lambda|)^{n-1}$ on vertical strips in the complex plane. Since $g(\lambda)$ is a Laplace transform, it is uniformly square integrable on vertical lines. Then (4.15) shows $f(\lambda)$ has the property that $f(\lambda)/(1 + |\lambda|)^n$ is uniformly square integrable on vertical lines in the complex plane. Using (4.8) again, together with (3.5 a,b) we see that there is a positive number $\tilde{M}$ such that for all real ξ

$$\left| \frac{\varphi(\xi + i\eta)\psi(\xi + i\eta)}{(1 + |\xi + i\eta|)^n} \right| \leq \frac{\tilde{M}Be^{\pi|\xi|}}{(1 + |\xi|)^{1/2}} .$$

Since g, being a Laplace transform, satisfies

$$|g(\xi + i\eta)| \leq \frac{M\hat{B}e^{\pi|\xi|}}{(1 + |\xi|)^{1/2}}$$

for some $\hat{B} > 0$ it follows that

$$\left| \frac{f(\xi + i\eta)}{1 + (\xi + i\eta)^n} \right| \leq \frac{\hat{M}\, e^{\pi|\xi|}}{(1 + |\xi|)^{1/2}} \tag{4.18}$$

for some $\hat{M} > 0$. Together with the uniform square integrability of $f(\lambda)/(1 + |\lambda|)^n$ on vertical lines, (4.18) shows, using Theorem 3.2, that

$$f(\lambda) = F(u,\lambda) \tag{4.19}$$

for some $u \in H^n[-\pi,\pi]'$. This completes the proof of Proposition 4.2.

Definition 4.3. Let $y \in H^{n+1}[-\pi,\pi]'$ and satisfy (4.7) while $x \in H^m[-\pi,\pi]$, $m \geq n$. The A-convolution of y with x is the element z described in Proposition 4.2. We write

$$z = (y * x)_A . \tag{4.20}$$

The remainder from the A-convolution of y with x (referred to as just the remainder when y,x,A are understood) is the element u described is Proposition 4.2.

It will be recognized that the decomposition of the product $\varphi(\lambda)\psi(\lambda)$ which we have just described generalizes one already familiar to users of the Laplace transform. If

$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ is a polynomial of degree n in λ and $p(D)$ is the corresponding polynomial in $D = \frac{d}{dt}$ and if $x, x'(t), \cdots, x^n(t)$, $t > 0$, have Laplace transforms, it is well known that

$$L(p(D)x,\lambda) = q(\lambda) + p(\lambda)L(x,\lambda), \tag{4.21}$$

where $q(\lambda)$ is a polynomial in λ of degree $\leq n-1$. If x is any linear combination of exponential functions, $\psi(\lambda) \equiv L(x,\lambda)$ and $g(\lambda) \equiv L(p(D)x,\lambda)$ are rational, i.e., meromorphic, functions. Letting $f(\lambda) = -q(\lambda)$, $p(\lambda) = \varphi(\lambda)$, $f(\lambda)$ and $\varphi(\lambda)$ are holomorphic and (4.21) becomes

$$g(\lambda) = \varphi(\lambda)\psi(\lambda) - f(\lambda)$$

in agreement with (4.15).

For reasons which will soon be apparent, our main interest lies in the case where $m = n$, so that y, satisfying (4.7), is an element of $H^{m+1}[-\pi,\pi]'$.

Theorem 4.4. *Let $x \in H^m[-\pi,\pi]$ and let x_A be its A-extension to $(-\infty,\infty)$. Let $y \in H^{m+1}[-\pi,\pi]'$ satisfy (4.7). Then*

$$(y * x)_A(t) = \underset{\|x - \xi_n\|_{H^m[-\pi,\pi]} \to 0}{\text{l.i.m.}} \langle y, \xi_{n,A}(t + \cdot)\rangle \tag{4.22}$$

in the sense that, for $\mathrm{Re}(\lambda) > \gamma^+$, $\mathrm{Re}(\lambda) < \gamma^-$, respectively

$$\lim_{n\to\infty} \|e^{-\lambda t}((y*x)_A(t) - \langle y, \xi_{n,A}(t + \cdot)\rangle)\|_{L^2[0,\infty)} = 0$$

$$\lim_{n\to\infty} \|e^{-\lambda t}((y*x)_A(t) - \langle y, \xi_{n,A}(t + \cdot)\rangle)\|_{L^2(-\infty,0]} = 0$$

for any sequence $\{\xi_n\} \subset \mathcal{D}(A)$ converging to x in $H^m[-\pi,\pi]$. If $x \in \mathcal{D}(A)$ then $(y * x)_A$ is continuous in t and

$$(y * x)_A(t) = \langle y, x_A(t + \cdot)\rangle . \tag{4.23}$$

Proof. We begin with $x \in \mathcal{D}(A)$. Then the right hand side of (4.23) is well defined and, since $x_A(t + \cdot)$ (the restriction of x_A to $[t - \pi, t + \pi]$) is continuous in $H^{m+1}[-\pi,\pi]$ when $x \in \mathcal{D}(A)$, it is continuous as a function of t. Exponential growth as $t \to \pm\infty$ is easily established and we may form the Laplace transform

$$\mathcal{L}\langle y, x_A(t+\cdot)\rangle = \begin{cases} \int_0^\infty e^{-\lambda t}\langle y, x_A(t+\cdot)\rangle\, dt, \\ \qquad \mathrm{Re}(\lambda) > \gamma^+ \\ \int_{-\infty}^0 e^{-\lambda t}\langle y, x_A(t+\cdot)\rangle\, dt, \\ \qquad \mathrm{Re}(\lambda) < \gamma^- . \end{cases} \tag{4.24}$$

From (3.42) we know that if $\varphi(\lambda)$ is the Fourier transform of y and $\psi_t(\lambda) = \psi_{t,0}(\lambda) + \psi_{t,1}(\lambda)$ is the A-Laplace transform of $x_A(t+\cdot)$,

$$\langle y, x_A(t+\cdot)\rangle = \frac{1}{2\pi i}\int_{\Gamma_0}\varphi(\lambda)\psi_{t,0}(\lambda)d\lambda + \frac{1}{2\pi i}\int_{\Gamma_1}\varphi(\lambda)\psi_{t,1}(\lambda)d\lambda .$$

Here and in the following, $\psi_{t,0}, \psi_{t,1}, \psi_0 \psi_1$ are computed relative to $H^{m+1}[-\pi,\pi]$, $H^{m+1}[-\pi,\pi]'$. Thus

$$\psi_{t,0}(\lambda) = \frac{\alpha_0}{\lambda} + \frac{\alpha_1}{\lambda^2} + \cdots + \frac{\alpha_m}{\lambda^{m+1}} ,$$

etc. Let the contours $\Gamma_0, \Gamma_1, C_0, C_1$ be arranged as in Figure 4.1 again. We have the formula (cf. (4.5))

$$\psi_t(\lambda) = \frac{1}{2\pi i}\int_{C_0}\frac{e^{\nu t}\psi_0(\nu)}{\lambda - \nu}\,d\nu + \frac{1}{2\pi i}\int_{C_1}\frac{e^{\nu t}\psi_1(\nu)}{\lambda - \nu}\,d\nu$$

(we have just replaced Γ_0, Γ_1 in (4.5) by C_0, C_1) valid for λ outside C_1, in particular on the contour Γ_1. From the fact that

$$\psi_0(\nu) = \frac{x(0)}{\nu} + \frac{x'(0)}{\nu^2} + \cdots + \frac{x^{(m)}(0)}{\nu^{m+1}}$$

it may be seen that the first integral has the form

$$\frac{1}{2\pi i}\int_{C_0}\frac{e^{\nu t}\psi_0(\nu)}{\lambda - \nu}\,d\nu = \frac{\beta_0}{\lambda} + \frac{\beta_1}{\lambda\lambda^2} + \cdots + \frac{\beta_m}{\lambda^{m+1}} \tag{4.25}$$

for constants $\beta_0, \beta_1, \cdots, \beta_m$. Since

$$\frac{1}{\lambda - \nu} = \frac{1}{\lambda} + \frac{\nu}{\lambda^2} + \cdots + \frac{\nu^m}{\lambda^{m+1}} + \frac{\nu^{m+1}}{\lambda^{m+1}}\frac{1}{\lambda - \nu}$$

we have

$$\frac{1}{2\pi i}\int_{C_1}\frac{e\ \ \psi_1(\nu)}{\lambda - \nu} = \frac{1}{2\pi i}\int_{C_1}\left(\frac{1}{\lambda} + \frac{\nu}{\lambda^2} + \cdots + \frac{\nu^m}{\lambda^{m+1}}\right)e^{\nu t}\psi_1(\nu)d\nu$$

$$+ \frac{1}{2\pi i}\int_{C_1}\frac{\nu^{m+1}}{\lambda^{m+1}}\frac{e^{\nu t}\psi_1(\nu)}{\lambda - \nu}\,d\nu .$$

Combining (4.25) and (4.26) we see that

$$\psi_{t,0}(\lambda) = \frac{1}{2\pi i} \int_{C_0} \frac{e^{\nu t}\psi_0(\nu)}{\lambda - \nu} d\nu + \frac{1}{2\pi i} \int_{C_1} \left(\frac{1}{\lambda} + \frac{\nu}{\lambda^2} + \cdots + \frac{\nu^m}{\lambda^{m+1}} \right) e^{\nu t}\psi_1(\nu) d\nu \ , \qquad (4.27)$$

$$\psi_{t,1}(\lambda) = \frac{1}{2\pi i} \int_{C_1} \frac{\nu^{m+1}}{\lambda^{m+1}} \frac{e^{\nu t}\psi_1(\nu)}{\lambda - \nu} d\nu \ . \qquad (4.28)$$

Then, noting that the first term on the right hand side of (4.26) extends, in the same form, inside C_1, and using (3.42) we have

$$\begin{aligned}
(y, x_A(t + \cdot)) &= \frac{1}{2\pi i} \int_{\Gamma_0} \varphi(\lambda)\psi_{t,0}(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} \varphi(\lambda)\psi_{t,1}(\lambda) d\lambda \\
&= \frac{1}{2\pi i} \int_{\Gamma_0} \varphi(\lambda) \Big[\frac{1}{2\pi i} \int_{C_0} \frac{e^{\nu t}\psi_0(\nu)}{\lambda - \nu} d\nu + \frac{1}{2\pi i} \int_{C_1} \left(\frac{1}{\lambda} + \cdots + \frac{\nu^m}{\lambda^{m+1}} \right) e^{\nu t}\psi_1(\nu) d\nu \Big] d\lambda \\
&\quad + \frac{1}{2\pi i} \int_{\Gamma_1} \varphi(\lambda) \Big[\frac{1}{2\pi i} \int_{C_1} \frac{\nu^{m+1}}{\lambda^{m+1}} \frac{e^{\nu t}\psi_1(\nu)}{\lambda - \nu} d\nu \Big] d\lambda \\
&= \frac{1}{2\pi i} \int_{C_0} e^{\nu t}\varphi(\nu)\psi_0(\nu) d\nu + \frac{1}{2\pi i} \int_{C_1} e^{\nu t}\psi_1(\nu) \Big[\frac{1}{2\pi i} \int_{\Gamma_0} \left(\frac{1}{\lambda} + \frac{\nu}{\lambda^2} + \cdots + \frac{\nu^m}{\lambda^{m+1}} \right) \varphi(\lambda) d\lambda \\
&\quad + \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\nu^{m+1}}{\lambda^{m+1}} \frac{\varphi(\lambda)}{\lambda - \nu} d\lambda \Big] d\nu \ . \qquad (4.29)
\end{aligned}$$

Let ν be any point on C_1. If $\tilde{\Gamma}$ is a rectangle centered at the origin, lying inside Γ_1 and containing ν in its interior, then, since the only singularity of $\frac{1}{\lambda} + \frac{\nu}{\lambda^2} + \cdots + \frac{\nu^m}{\lambda^{m+1}}$ is at zero and the only singularities of $\frac{\nu^{m+1}}{\lambda^{m+1}} \frac{\varphi(\lambda)}{\lambda - \nu}$ lie inside $\tilde{\Gamma}$, we have

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\Gamma_0} \left(\frac{1}{\lambda} + \frac{\nu}{\lambda^2} + \cdots + \frac{\nu^m}{\lambda^{m+1}} \right) \varphi(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\nu^{m+1}}{\lambda^{m+1}} \frac{\varphi(\lambda)}{\lambda - \nu} d\lambda \\
&= \frac{1}{2\pi i} \int_{\tilde{\Gamma}} \frac{\varphi(\lambda)}{\lambda - \nu} d\lambda = \varphi(\nu)
\end{aligned}$$

and (4.29) gives

$$(y, x_A(t + \cdot)) = \frac{1}{2\pi i} \int_{C_0} e^{\nu t}\varphi(\nu)\psi_0(\nu) d\nu + \frac{1}{2\pi i} \int_{C_1} e^{\nu t}\varphi(\nu)\psi_1(\nu) d\nu \ . \qquad (4.30)$$

Computing the Laplace transform for λ outside C_1 (λ to the right of the right had branch of C_1 shown here) we find using Definition 4.3 that

$$\int_0^\infty e^{-\lambda t} \langle y, x_A(t+\cdot)\rangle \, dt$$

$$= \int_0^\infty e^{-\lambda t}\left[\frac{1}{2\pi i}\int_{C_0} e^{\nu t}\varphi(\nu)\psi_0(\nu)\,d\nu + \frac{1}{2\pi i}\int_{C_1} e^{\nu t}\varphi(\nu)\psi_1(\nu)\,d\nu\right]$$

$$= \frac{1}{2\pi i}\int_{C_0} \frac{\varphi(\nu)\psi_0(\nu)}{\lambda-\nu}\,d\nu + \frac{1}{2\pi i}\int_{C_1}\frac{\varphi(\nu)\psi_1(\nu)}{\lambda-\nu}\,d\nu \tag{4.31}$$

$$= L((y * x)_A, \lambda) .$$

Thus

$$L(\langle y, x_A(t+\cdot)\rangle, \lambda) = L((y * x)_A, \lambda)$$

and we conclude by the familiar uniqueness theorem for the Laplace transform, that (4.23) is valid.

Now let $\{\xi_n\} \subseteq \mathcal{D}(A)$ with

$$\lim_{n\to\infty} \|x - \xi_n\|_{H^m[-\pi,\pi]} = 0$$

for some $x \in H^m[-\pi,\pi]$. For each n we have

$$(y * \xi_n)_A(t) = \langle y, \xi_{n,A}(t+\cdot)\rangle$$

with Laplace transform (cf. Definition 4.3 and (4.9))

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\varphi(\nu)\psi_{n,0}(\nu)}{\lambda-\nu}\,d\nu + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\varphi(\nu)\psi_{n,1}(\nu)}{\lambda-\nu}\,d\nu .$$

Now

$$\psi_{n,0}(\nu) = \frac{\xi_n(0)}{\nu} + \frac{\xi_n'(0)}{\nu^2} + \cdots + \frac{\xi_n^{(m)}(0)}{\nu^{m+1}}$$

$$\psi_{n,1}(\nu) = \frac{1}{\nu^{m+1}}\int_0^\infty e^{-\nu t}\xi_A^{(m+1)}(t)\,dt.$$

We regroup these as

$$\tilde{\psi}_{n,0}(\nu) = \frac{\xi_n(0)}{\nu} + \frac{\xi_n'(0)}{\nu^2} + \cdots + \frac{\xi_n^{(m-1)}(0)}{\nu^m}$$

$$\tilde{\psi}_{n,1}(\nu) = \frac{\xi_n^{(m)}(0)}{\nu^{m+1}} + \frac{1}{\nu^{m+1}}\int_0^\infty e^{-\nu t}\xi_{n,A}^{(m+1)}(t)\,dt$$

$$= \frac{1}{\nu^m}\int_0^\infty e^{-\nu t}\xi_{n,A}^{(m)}(t)\,dt.$$

Observing that

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\varphi(\nu)\left(\dfrac{\xi_n^{(m)}(0)}{\nu^{m+1}}\right)}{\lambda-\nu}\,d\nu=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\varphi(\nu)\left(\dfrac{\xi_n^{(m)}(0)}{\nu^{m+1}}\right)}{\lambda-\nu}\,d\nu$$

we have

$$L((y*\xi_n)_A,\lambda)=\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\varphi(\nu)\tilde{\psi}_{n,0}(\nu)}{\lambda-\nu}\,d\nu+\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\varphi(\nu)\tilde{\psi}_{n,1}(\nu)d\nu}{\lambda-\nu}\ .$$

But (defining $\tilde{\psi}_0,\tilde{\psi}_1$ with reference to $H^m[-\pi,\pi]$ now) Definition 4.3 and (4.9) give

$$L((y*x)_A,\lambda)=\frac{1}{2\pi i}\int_{\Gamma_0}\frac{\varphi(\nu)\tilde{\psi}_{n,0}(\nu)}{\lambda-\nu}\,d\nu+\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\varphi(\nu)\tilde{\psi}_1(\nu)}{\lambda-\nu}\,d\nu\ .$$

Let us write

$$(y*\xi_n)_A(t)=z_n(t),\quad (y*x)_A(t)=z(t).$$

Then, since $z_n(t)$, $z(t)$ have the indicated transforms

$$z(t)-z_n(t)=\frac{1}{2\pi i}\int_{\Gamma_0}e^{\nu t}\varphi(\nu)(\tilde{\psi}_0(\nu)-\tilde{\psi}_{n,0}(\nu))d\nu$$

$$+\frac{1}{2\pi i}\int_{\Gamma_1}e^{\nu t}\varphi(\nu)(\tilde{\psi}_1(\nu)-\tilde{\psi}_{n,1}(\nu))d\nu\ .$$

The proof of (4.22) then follows the same lines as in (4.17), ff. and the proof is complete.

We complete this section with a result which forms the core of our work in Section 5.

<u>Theorem 4.5.</u> <u>If</u> η <u>is the generating distribution associated with the neutral generator</u> A <u>on</u> $H^m[-\pi,\pi]$, <u>if</u> $x \in H^m[-\pi,\pi]$ <u>and</u> u <u>is the remainder from the</u> A-<u>convolution of</u> η <u>with</u> x, <u>then</u>

$$\psi(\lambda)=L_A(x,\lambda)=\frac{F(u,\lambda)}{F(\eta,\lambda)}\ \left(=\frac{f(\lambda)}{(\lambda)}\right)\ . \tag{4.32}$$

<u>Remark</u> For our work here we will make the assumption that $\eta \in H^m[-\pi,\pi]'$ satisfies (4.8) but (4.32) is true under more general circumstances.

<u>Proof</u>. It is only necessary to observe that for $\xi_n \in \mathcal{D}(A)$, $\langle \eta, \xi_{n,A}(t+\cdot)\rangle \equiv 0$. Hence $(\eta * x)_A = 0$ almost everywhere and $g(\lambda)=L((\eta*x)_A,\lambda)\equiv 0$. Then (4.32) follows from (4.15).

5. Exponential Bases from Neutral Groups.

Armed with the transform theory which we developed in Section 4, we are now prepared to discuss Question B posed in Section 2. We cannot give complete answers to this question but we can make some beginnings. We shall suppose that A is a neutral generator on $H^m[-\pi,\pi]$ with domain $\{x \in H^{m+1}[-\pi,\pi] \mid \langle x,\eta\rangle = 0\}$, η being the generating distribution in $H^m[-\pi,\pi]'$. We shall suppose that $\varphi(\lambda) = F(\eta,\lambda)$ satisfies (4.8) with $n = m$. We hope to devote another paper to some significant cases wherein (4.7), (4.8) do not apply. We will content ourselves here with the statement that it is useless to try to prove that every neutral generator A corresponds to a generating distribution η for which (4.8) is valid. By use of the gamma function one can easily construct η (via $F(\eta,\lambda)$) for which A generates a group on $H^m[-\pi,\pi]$ but (either part) of (4.8) fails.

The work done in Section 4 shows that η satisfying (4.7) yields a map

$$\left.\begin{aligned} &B: H^m[-\pi,\pi] \to H^m[-\pi,\pi]' , \\ &Bx = u , \end{aligned}\right\} \qquad (5.1)$$

where u is the remainder from $(\eta * x)_A$. Following a comment by Mihailov in [18] we will call B the Bary (after N. K. Bary) operator associated with η. It has the properties set forth in

<u>Theorem 5.1.</u> <u>If</u> B <u>is the Bary operator associated with the generating distribution</u> η <u>of a neutral group on</u> $H^m[-\pi,\pi]$ <u>and if</u> η <u>satisfies</u> (4.8) <u>with</u> $n = m$, <u>then</u>

(i) B <u>is one to one and bounded.</u>

(ii) B <u>is onto in the case</u> $m(=n) = 0$.

(iii) <u>Let</u> λ_k <u>be a zero of</u> $F(\eta,\lambda)$ <u>of multiplicity</u> m_k. <u>Then</u> B <u>maps the subspace of</u> $H^m[-\pi,\pi]$ <u>spanned by the generalized exponentials</u> $e^{\lambda_k t}, te^{\lambda_k t}, \cdots, (t^{m_k-1}/(m_k-1)!)e^{\lambda_k t}$ <u>onto the dual subspace of</u> $H^m[-\pi,\pi]'$ <u>spanned by elements</u> $q_{k,\ell}$, $\ell = 1,2,\cdots,m_k$ <u>which have the properties</u> $(p_{k,\ell}(t) = (t^{\ell-1}/(\ell-1)!)e^{\lambda_k t})$

$$\langle p_{k,\ell}, q_{k,\hat{\ell}} \rangle = \delta_{\ell,\hat{\ell}} = \begin{cases} 1 & \ell = \hat{\ell} \\ 0 & \ell \neq \hat{\ell} \end{cases} \tag{5.2}$$

$$\langle p_{\tilde{k},\tilde{\ell}}, q_{k,\ell} \rangle = 0 , \quad \ell = 1,2,\cdots,m_{\tilde{k}} \tag{5.3}$$

<u>if</u> $\lambda_{\tilde{k}}$ <u>is a different zero of</u> $F(\eta,\lambda)$ <u>of multiplicity</u> $m_{\tilde{k}}$.

<u>Proof.</u> That B is one to one is immediate: if $u = 0$ in $H^m[-\pi,\pi]'$ then (4.32) shows that $L_A(x,\lambda) = 0$ which implies, via the uniqueness theorem for the Laplace transform, that $x_A(t) \equiv 0$ and hence $x = 0$ in $H^m[-\pi,\pi]$. The boundedness of B is an easy strengthening of the statement $u \in H^{n(=m)}[-\pi,\pi]'$ in Proposition 4.2.

To show that B is onto when $m(=n) = 0$, let

$$\psi(\lambda) = F(u,\lambda)/F(\eta,\lambda).$$

From Theorem 3.2 $\psi(\lambda)$ is square integrable on vertical lines in the complex plane and, with Γ_1 defined as before

$$\theta(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\psi(\nu)}{\lambda - \nu} \, d\nu \tag{5.4}$$

converges for λ outside Γ_1 . We will show that $\theta(\lambda) = \psi(\lambda)$. Using Theorem 3.2, (4.8), for $|Re(\lambda)|$ sufficiently large

$$|\psi(\lambda)| = \left|\frac{F(u,\lambda)}{F(\eta,\lambda)}\right| \leq \frac{M}{(1 + |Re(\lambda)|)^{1/2}} \tag{5.5}$$

Let contours Γ_R^+ , Γ_R^- be constructed as shown in Figure 5.1, the semi-circular parts of each being denoted by Σ_R^+ , Σ_R^- .

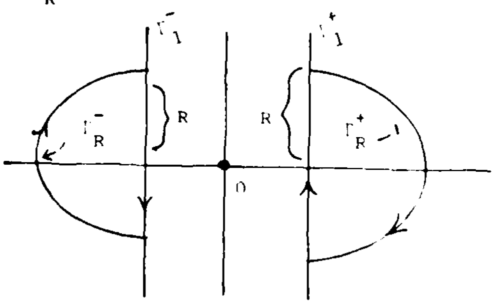

Figure 5.1

For λ inside Γ_R^+ (i.e., for R sufficiently large)

$$\frac{1}{2\pi i} \int_{\Gamma_R^+} \frac{\psi(\nu)}{\lambda - \nu} \, d\nu = \psi(\lambda), \quad \frac{1}{2\pi i} \int_{\Gamma_R^-} \frac{\psi(\nu)}{\lambda - \nu} \, d\nu = 0 . \tag{5.6}$$

We easily estimate, using (5.5),

$$\left|\frac{1}{2\pi i}\int_{\Sigma_R^+(\Sigma_R^-)} \frac{\psi(\nu)}{\lambda-\nu}\,d\nu\right| \leq \frac{\hat{M}}{1+R^{1/2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{(\cos\theta)^{1/2}} \to 0$$

as $R \to \infty$, $\hat{M}$ being a convenient constant. It follows that

$$\lim_{R\to\infty}\left(\frac{1}{2\pi i}\int_{\Gamma_R^+}\frac{\psi(\nu)}{\lambda-\nu}\,d\nu + \frac{1}{2\pi i}\int_{\Gamma_R^-}\frac{\psi(\nu)}{\lambda-\nu}\,d\nu\right) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\psi(\nu)}{\lambda-\nu}\,d\nu$$

and (5.4), (5.6) give

$$\psi(\lambda) = \theta(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\psi(\nu)}{\lambda-\nu}\,d\nu .$$

Thus $\psi(\lambda)$ is the Laplace transform of

$$\xi(t) = \text{l.i.m.}\ \frac{1}{2\pi i}\int_{\Gamma_1} e^{\nu t}\psi(\nu)\,d\nu .$$

Since $F(\eta,\lambda)\psi(\lambda) = F(u,\lambda)$ is entire, it follows that $L((\eta * \xi),\lambda) = 0$ and hence $\eta * \xi = 0$ so that $\xi(t) = x_A(t)$ if $x(t)$ is the restriction of ξ to $[-\pi,\pi]$. Then $u = B(x)$, $x \in L^2[-\pi,\pi]$ and B is onto for the case $m(=n) = 0$. This result does not extend to $m[=n] > 0$.

To establish (iii) it is enough to observe that if $x \in H^m[-\pi,\pi]$ is a linear combination of the generalized exponentials $(t^{\ell-1}/(\ell-1)!)e^{\lambda_k t}$ then $L_A(x,\lambda)$ is a polynomial in $(\lambda-\lambda_k)^{-1}$ of the form

$$L_A(x,\lambda) = \frac{\gamma_1}{\lambda-\lambda_k} + \frac{\gamma_2}{(\lambda-\lambda_k)^2} + \cdots + \frac{\gamma_m}{(\lambda-\lambda_k)^m} .$$

Then

$$F(u,\lambda) = F(\eta,\lambda)L_A(x,\lambda)$$

is either nonvanishing at $\lambda - \lambda_k$ (if $\gamma_m \neq 0$) or else has a zero of order less than m there, assuming $x \neq 0$. On the other hand, a zero $\lambda_{\tilde{k}} \neq \lambda_k$ of multiplicity $m_{\tilde{k}}$ of $F(\eta,\lambda)$ remains a zero of multiplicity $m_{\tilde{k}}$ of (u,λ). This shows that B maps the subspace spanned by the $p_{k,\ell}$ into the subspace spanned by the $q_{k,\ell}$, which is the subspace of $H^m[-\pi,\pi]'$ whose elements y have F-transforms $F(y,\lambda)$ which have zeros of multiplicity $m_{\tilde{k}}$ at $\lambda_{\tilde{k}} \neq \lambda_k$. About $\lambda = \lambda_k$ we have

$$(u,\lambda) = \alpha_0 + \alpha_1(\lambda-\lambda_k) + \cdots + \alpha_{m_k-1}(\lambda-\lambda_k)^{m_k-1} + \cdots$$

and it may be seen that the finite dimensional linear map $(\gamma_1,\cdots,\gamma_{m_k}) \to (\alpha_0,\cdots,\alpha_{m_k-1})$ is nonsingular. Hence B maps the subspace spanned by the $p_{k,\ell}$ onto the subspace spanned by the $q_{k,\ell}$ and the proof is complete.

We turn next to the question of completeness. Let $\{\lambda_k | k \in K\}$ denote the set of zeros of $F(\eta,\lambda)$ and let $\{m_k | k \in K\}$ be the corresponding set of multiplicities. Here K is any convenient countable index set. A condition for completeness of the set

$$\left\{ \frac{t^{\ell-1}}{(\ell - 1)!} e^{\lambda_k t} \mid k \in K, \quad \ell = 1,2,\cdots,m_k \right\} \tag{5.7}$$

is given by Levin in [13]. The proof is based on use of the indicator diagram of $F(\eta,\lambda)$ and an assumption about the growth of that function similar to (4.8).

When η is the generating distribution defining the domain of a neutral generator, A, with $F(\eta,\lambda)$ satisfying (4.8), these considerations may be replaced with an argument using the representation (4.32) together with the minimum modulus theorem (see [29], e.g.).

<u>Theorem 5.2.</u> <u>Let</u> η <u>be the generating distribution associated with the neutral generator</u> A <u>on</u> $H^m[-\pi,\pi]$ <u>with</u> $F(\eta,\lambda)$ <u>satisfying</u> (4.8). <u>Let</u>

$$E_\eta = \{(t^{\ell-1}/(\ell-1)!)e^{\lambda_k t} \mid k \in K, \quad \ell = 1,2,\cdots,m_k\} \tag{5.8}$$

<u>be the set of exponentials associated with the zeros</u> λ_k, <u>of multiplicity</u> m_k, <u>of</u> $F(\eta,\lambda)$. <u>Then</u> E_η <u>is complete in</u> $H^m[-\pi,\pi]$.

<u>Proof.</u> It is well known from entire functions theory that $F(\eta,\lambda)$ must have infinitely many zeros. Select m of these; call them $\sigma_1,\sigma_2,\cdots,\sigma_m$. Let

$$p(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k)$$

and let η_0 be the element of $L^2[-\pi,\pi]$ whose F transform is $F(\eta,\lambda)/p(\lambda)$. Arguments similar to those given in Section 2 show that E_η is complete in $H^m[-\pi,\pi]$ if and only if, with μ_k the multiplicity of σ_k,

$$E_{\eta_0} = E_\eta - \{t^{\mu_k-1}/(\mu_k-1)!)w^{\sigma_k t} \mid k = 1,2,\cdots,m\}$$

is complete in $L^2[-\pi,\pi]$. As we have seen from Section 2 the operator A_0 (differentiation on H^1) with domain described by $\langle x,\eta_0 \rangle = 0$ generates a group on $L^2[-\pi,\pi]$ if and only if A, with domain designated by $\langle x,\eta \rangle = 0$, generates a group on $H^m[-\pi,\pi]$.

If E_{η_0} were not complete in $L^2[-\pi,\pi]$ there would exist a non-zero element $u \in L^2[-\pi,\pi] = L^2[-\pi,\pi]'$ such that $\langle u,p \rangle = 0$ for each p of the form

$$p(t) = (t^{\ell-1}/\ell-1)!e^{\lambda_k t} \in E_{\eta_0} .$$

Since Theorem 3.2 gives

$$\langle u,p \rangle = \frac{1}{2\pi i} \int_{\Gamma_1} F(u,\lambda) \frac{1}{(\lambda - \lambda_k)^{\ell}} d\lambda$$

$$= \mathrm{Res}\left[F(u,\lambda) \frac{1}{(\lambda - \lambda_k)^{\ell}} \right]_{\lambda = \lambda_k}$$

we conclude that $F(u,\lambda)$ has zeros at the zeros, λ_k, of $F(\eta_0,\lambda)$, of at least equal multiplicity. Hence $F(u,\lambda)/F(\eta,\lambda)$ is entire. Since B is onto for $m = 0$ ($F(\eta_0,\lambda)$ satisfies (4.8) with $m = 0$) we conclude that there is an element $x \in L^2[-\pi,\pi]$ such that

$$L_{A_0}(x,\lambda) = \frac{F(u,\lambda)}{F(\eta_0,\lambda)} .$$

But then, as we have just noted, $L_{A_0}(x,\lambda)$ is entire. From this we can show that $L_{A_0}(x,\lambda) = 0$, implying $x = 0$.

First of all, the fact that $F(\eta_0,\lambda)$ is of order 1 and type π implies (see, e.g. [13]) that the zeros of $F(\eta_0,\lambda)$, counting multiplicity must be such that, with $N(R)$ denoting the number of zeros of $F(\eta_0,\lambda)$ inside the disc of radius R in the complex plane

$$N(R) \leq KR \tag{5.9}$$

for some positive constant K.

We know, since A_0 generates a group, that we can find a contour $\Gamma_1 = \Gamma_1^+ \cup \Gamma_1^-$ as shown in Figure 5.2 containing the zeros of $F(\eta_0,\lambda)$ in its interior. From the density of the zeros as described by (5.9), it follows that we may construct paths C_r^+, C_r^-, (the segments of $\mathrm{Im}(\lambda) = r$, $\mathrm{Im}(\lambda) = -r$ lying between Γ_1^+ and Γ_1^-) for a sequence of values of $r = r_n$ tending to ∞, such that

$$\mathrm{dis}(\Lambda, C_r^{\pm}) \geq \varepsilon , \quad \text{all} \quad r = r_n ,$$

for some fixed $\varepsilon > 0$, Λ denoting the set of zeros of $F(\eta_0,\lambda)$. Using the order and type of $F(\eta_0,\lambda)$ together with the minimum modulus theorem (see, e.g. [29], Chapter VIII) we see that

$$|F(\eta,\lambda)| \geq \mu\, e^{-|\lambda|(\pi + \varepsilon)} , \quad \lambda \in C^{\pm}_{r_n} ,$$

where μ is some positive number and ε is an arbitrarily small positive number. Then, since $F(u,\lambda)$ is bounded in the interior of Γ_1

$$L_A(x,\lambda) \leq \nu\, e^{|\lambda|(\pi + \varepsilon)} , \quad \lambda \in C^{\pm}_{r_n} .$$

The result (5.9) also implies that the $C^{\pm}_r$ can be selected so that the r_k grow at rates which may be bounded above and below by positive multiples of k. Then, since $F(u,\lambda)/F(\eta_0,\lambda)$ is entire it follows that, in the interior of Γ_1,

$$L_{A_0}(x_0,\lambda) \leq D\, e^{|\lambda|(\pi + \varepsilon)} .$$

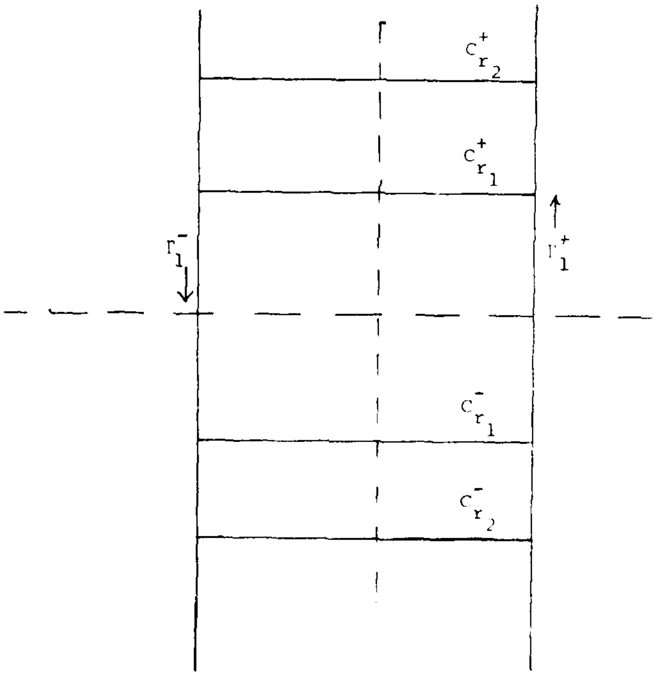

Figure 5.2

Combined with the fact that $L_A(x_0,\lambda)$, being the A-Laplace transform of an element $x \in L^2[-\pi,\pi]$, must be bounded on Γ_1, we may use the Phragmen-Lindelof theorem for a strip (see, e.g. [9], Chapter 18) to see that $L_{A_0}(x,\lambda)$ must be bounded by a constant in the region bounded by the contour Γ_1. But $L_A(x,\lambda)$ is uniformly bounded on vertical lines outside Γ_1 and

$$\lim_{|\mathrm{Re}(\lambda)|\to\infty} L_A(x,\lambda) = 0$$

so $L_A(x,\lambda)$, and hence x, must be zero. It follows then that $u = 0$, contradicting our earlier assumption. This completes the proof of Theorem 5.2.

Since the existence of biorthogonal elements in $H^m[-\pi,\pi]'$ is already assured by Theorem 5.1, we know that E_η, described by (5.8), forms a strongly independent basis for $H^m[-\pi,\pi]$. This, however, is not the same thing as showing that each element $x \in H^m[-\pi,\pi]$ can be developed as a series in the elements of E_η, convergent in $H^m[-\pi,\pi]$. For this we must have properties of $F(\eta_0,\lambda)$ which are stronger than those which may be inferred from the minimum modulus theorem.

It is not necessary to state and prove a separate theorem because we already have Theorem 3.9. What is necessary is to establish conditions sufficient to guarantee that the condition (3.52) is staisfied. The representation (4.32) can be very useful here. We begin with the case $m = 0$.

Proposition 5.3. Let $m = 0$ and suppose that $\eta \in H^1[-\pi,\pi]'$ is the generating distribution corresponding to a neutral generator A (of the form (2.9) with $m = 0$) on $L^2[-\pi,\pi]$. If the paths C_j, C_{-j} in the proof of Theorem 3.9 can be selected so that

$$|F(\eta,\lambda)| \geq \delta, \quad \lambda \in C_j, \quad \lambda \in C_{-j} \tag{5.10}$$

for all j and some fixed $\delta > 0$, then (3.52) follows and the expansion (3.53) follows.

Proof. It is enough to note that $F(u,\lambda)$, being the Fourier transform of an element $u \in L^2[-\pi,\pi]$ has the property

$$\lim_{j\to\infty} \sup_{\lambda \in C_j(C_{-j})} \{|F(u,\lambda)|\} = 0 \tag{5.11}$$

by virtue of the Riemann Lebesque theorem.

Corollary 5.4. In the case $m = 0$, (5.10), and hence (3.52), is true if $F(\eta,\lambda)$ is uniformly almost periodic in a strip

$$|\mathrm{Re}\ \lambda| < \rho$$

which includes the contour Γ_1.

Proof. This is clear from the definition of almost periodicity. See the related result in [4].

An example which one might cite here is

$$\eta = \delta_{(\pi)} + c_0\delta_{(-\pi)} + \sum_{k=1}^{\infty} c_k\,\delta_{(\sigma_k)}$$

where the σ_k are distinct points in $(-\pi,\pi)$, $c_0 \neq 0$ and $\{c_k\} \in \ell^1$. The almost periodicity of (η,λ) follows from results in [4], [13].

Proposition 5.5. Let C_j, C_{-j} be as in Propositin 5.3 and Theorem 3.9. Then, for an arbitrary integer $m > 0$ (3.52) remains true if $F(\eta,\lambda)$ satisfy (4.8) and (5.10) is replaced by

$$|F(\eta,\lambda)| > \delta|\xi|^m, \quad \lambda \in C_j, \quad \lambda \in C_{-j} \tag{5.12}$$

for all j. The expansion results 3.9 follow in $H^m[-\pi,\pi]$.

Proof. Let x_A be the A-extension of x and let $L_A(x,\lambda) = L(x_A,\lambda)$ as before. Let $\sigma_1,\sigma_2,\cdots,\sigma_m$ be m zeros (possibly including multiplicity, of $F(\eta,\lambda)$. Let

$$p(\lambda) = \prod_{k=1}^{m} (\lambda - \sigma_k), \quad p(D) = \prod_{k=1}^{m} (D - \sigma_k I).$$

Let

$$z = p(D)x_A.$$

Then for $\mathrm{Re}(\lambda) > \rho$, $e^{-\lambda t}z(t)$ $L^2[0,\infty)$, for $\mathrm{Re}(\lambda) < -\rho$, $e^{-\lambda t}L^2(-\infty,0]$ and we conclude that $L(z,\lambda) \in L^2(\Gamma_1)$, assuming Γ_1 lies in $|\mathrm{Re}(\lambda)| > \rho$. The familiar computation from ordinary differential equations shows that

$$L(x_A,\lambda) = \frac{q(\lambda)}{p(\lambda)} + \frac{L(z,\lambda)}{p(\lambda)} \tag{5.13}$$

where $q(\lambda)$ is a polynomial in λ of degree $\leq m-1$. Since the zeros of $p(\lambda)$ are zeros of $F(\eta,\lambda)$ we have

$$F(\eta,\lambda) = p(\lambda)F(\eta_0,\lambda)$$

where $F(\eta_0,\lambda)$ is (as a result of the assumption (4.8) on $F(\eta,\lambda)$) bounded and bounded away from zero on Γ_1. As seen earlier, η_0 is the generating distribution for a neutral group on $L^2[-\pi,\pi]$. Now (4.32) gives, with (5.13),

$$p(\lambda)F(\eta_0,\lambda)\left(\frac{q(\lambda)}{p(\lambda)} + \frac{L(z,\lambda)}{p(\lambda)} \right) = F(u,\lambda)$$

whence

$$L(z,\lambda) = \frac{F(u,\lambda) - q(\lambda)\; F(\eta_0,\lambda)}{F(\eta_0,\lambda)} . \tag{5.14}$$

Since $L(z,\lambda)$ is in $L^2(\Gamma_1)$ and $F(\eta_0,\lambda)$ is bounded and bounded away from zero on Γ_1, we conclude that the entire function $F(u,\lambda) - q(\lambda)F(\eta_0,\lambda) \in L^2(\Gamma_1)$ and, being of order 1 and type π, must be the F transform of an element $w \in L^2[-\pi,\pi]$; thus

$$F(u,\lambda) = F(w,\lambda) + q(\lambda)F(\eta_0,\lambda) \tag{5.15}$$

where $q(\lambda)$, as indicated earlier, is a polynomial in λ of degree $\leq m-1$.

Now (5.14) gives

$$L(z,\lambda) = \frac{F(w,\lambda)}{F(\eta_0,\lambda)}$$

and (5.13) then gives

$$L_A(x,\lambda) = L(x_A,\lambda) = \frac{q(\lambda)}{p(\lambda)} + \frac{1}{p(\lambda)}\frac{L(w,\lambda)}{F(\eta_0,\lambda)} = \frac{q(\lambda)}{p(\lambda)} + \frac{L(w,\lambda)}{F(\eta,\lambda)} \equiv \tilde{\psi}_1(\lambda) + \tilde{\psi}_1(\lambda).$$

The functions $\tilde{\psi}_0(\lambda)$ and $\tilde{\psi}_1(\lambda)$ can be used in the same way as $\psi_0(\lambda)$, $\psi_1(\lambda)$ were in Theorem 3.9. The form

$$\tilde{\psi}_1(\lambda) = \frac{L(w,\lambda)}{F(\eta,\lambda)}$$

enables one to use the Riemann-Lebesgue theorem on $L(w,\lambda)$ together with the assumptions (5.12) on $F(\eta,\lambda)$ to see that (3.52) is true with ψ_1 replaced by $\tilde{\psi}_1$. The results then follow immediately.

We remark that (5.14) shows why the map B is not onto for $m > 0$. Its range is contained in the subspace of $H^m[-\pi,\pi]'$ spanned by $L^2[-\pi,\pi]$ and the m distributions $\eta_0, \delta'_{(0)} * \eta_0, \cdots, \delta^{(m-1)}_{(0)} * \eta_0$. It is noteworthy that the elements of $H^m[-\pi,\pi]'$ biorthogonal to the generalized exponentials in E_η lie in this subspace of $H^m[-\pi,\pi]'$.

We will conclude this report with a short discussion of the application of our theory to the study of uniform bases of exponentials. We confine attention to the case $m = 0$ and we make the following assumptions:

(i) η_0 is the generating distribution associated with a neutral generator A on $L^2[-\pi,\pi]$;

(ii) $F(\eta_0,\lambda)$ satisfies (4.8) with $n(=m) = 0$;

(iii) the zeros, λ_k, $k \in K$, of $F(\eta_0,\lambda)$ are uniformly separated:

$$|\lambda_j - \lambda_k| \geq D , \quad j \neq k , \tag{5.16}$$

where D is a fixed positive number.

(iv) Let

$$\Lambda = \{\lambda_k | \ k \in \Lambda\} .$$

We assume there is a positive number M such that for every complex λ

$$|F(\eta_0\lambda)| \geq M \ \mathrm{dis}(\lambda,\Lambda). \tag{5.17}$$

We begin by establishing

Proposition 5.6. Let η_0 be the generating distribution associated with a neutral generator A_0 on $L^2[-\pi,\pi]$ with Fourier transform $\varphi(\lambda) = F(\eta_0,\lambda)$ satisfying (4.8). Let $u \in L^2[-\pi.\pi]$ and let

$$\vartheta(\lambda) = F(u,\lambda) .$$

Assuming (i) - (iv) satisfied,

$$\vartheta(\lambda) = \sum_{k \in K} \left(\frac{\vartheta(\lambda_k)}{\varphi'(\lambda_k)} \right) \frac{\varphi(\lambda)}{\lambda - \lambda_k} \tag{5.18}$$

for every complex $\lambda \notin \Lambda$, the convergence being uniform on compact sets in the complex plane which do not meet Λ .

Proof. Let $x \in L^2[-\pi,\pi]$ with A_0-Laplace transform $L_{A_0}(x,\lambda)$ be such that, from Theorem 4.5,

$$\psi(\lambda) = \frac{\vartheta(\lambda)}{\varphi(\lambda)} . \tag{5.19}$$

Let $\Gamma_1 = \Gamma_1^+ \cup \Gamma_1^-$ be constructed as in Figure 5.1 so that λ and Λ lie in its interior.

Form the integral

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\psi(\nu)}{\lambda-\nu}\,d\nu\ .$$

Following the argument applied to (5.4), we can show this integral to be zero for $\lambda \in \mathrm{Int}(\Gamma_1)$. Then, using (5.19),

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\vartheta(\lambda)\,d\nu}{\varphi(\nu)(\lambda-\nu)} = 0\ .$$

Constructing cross paths C_j, C_{-j} from Γ_1^+ to Γ_1^- of uniformly bounded length, at a uniform distance from Λ (which (5.16) permits, we can use (5.17) to see that

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\psi(\nu)}{\lambda-\nu}\,d\nu = \sum \mathrm{Res}(\psi(\nu)/\lambda-\nu)$$

the sum taken over the poles of $\psi(\nu)/\lambda-\nu)$, inside Γ_1. From (5.19) the residue at $\nu=\lambda$ is $-\vartheta(\lambda)/\varphi(\lambda)$ while at each $\lambda_k \in \Lambda$ the residue is $\vartheta(\lambda_k)/[\varphi'(\lambda_k)(\lambda-\lambda_k)]$. Since the sum of the residues is zero, (5.18) follows. That the convergence is uniform on *compact sets not meeting* Λ *(this last condition can be* dispensed with) follows from the fact that $\int_{C_{\pm j}}(\psi(\nu)/\lambda-\nu)d\nu$ tends uniformly to zero as $C_{\pm j}\to\infty$ for λ so restricted.

<u>Corollary 5.7.</u> <u>Under the hypotheses of Proposition</u> 5.6, <u>with</u> $\psi(\lambda)$ <u>the A-Laplace</u> <u>transform of</u> $x \in L^2[-\pi,\pi]$,

$$\psi(\lambda) = \sum_{k\,\in K}\frac{\vartheta(\lambda_k)}{\varphi'(\lambda_k)}\,\frac{1}{\lambda-\lambda_k}\ . \tag{5.20}$$

<u>Remark.</u> This is, of course, essentially Corollary 5.4 and Theorem 3.9 again, with the coefficients identified more explicitly.

<u>Theorem 5.8.</u> <u>Under the hypotheses of Proposition</u> 5.6 (<u>including assumptions</u> (i)-(iv)), <u>the</u> $e^{\lambda_k t}$, $k \in K$, <u>form a uniform basis for</u> $L^2(-\pi,\pi]$.

<u>Proof.</u> We begin by noting that if we multiply $1/\varphi(\lambda) = 1/F(\eta_0,\lambda)$ by $\lambda-\lambda_k$ and use (5.17) we obtain

$$|\varphi'(\lambda_k)| \geq M\ ,\quad k \in K\ .$$

The results presented already in this section show the $e^{\lambda_k t}$, $k \in K$, to be complete in $L^2[-\pi,\pi]$ and, since the functions $q_k(t) \in L^2[-\pi,\pi]$ whose Fourier transforms are the functions

$$\frac{\varphi(\lambda_k)}{\varphi'(\lambda_k)(\lambda - \lambda_k)}, \quad k \in K,$$

are readily seen to satisfy, with $p_k(t) = e^{\lambda_k t}$,

$$(p_k, q_\ell)_{L^2[-\pi,\pi]} = \delta_{k\ell}, \qquad k, \ell \in K$$

the strong independence of the $e^{\lambda_k t}$, $k \in K$, *is assured. Consider sequences*

$$\{a_k \mid k \in K\} \tag{5.22}$$

in ℓ^2 with the usual norm. Define

$$T : \{a_k\} \to \sum_{k \in K} a_k e^{\lambda_k t}.$$

The domain of T is initially sequences (5.22) with all but finitely many a_k equal to zero. Then we extend T in the obvious way to all $\{a_k\}$ for which $\sum_{k \in K} a_k e^{\lambda_k t}$ is convergent in $L^2[-\pi,\pi]$. Then T is densely defined and one to one with dense range in $L^2[-\pi,\pi]$. The adjoint map

$$T^* : \sum_{k \in K} b_k q_k \to \{b_k\}$$

is again easily seen to have dense domain and range and to be one to one.

Now, in fact, the operators T and T^* are both bounded. This follows from a result, proved in [5] and [22], for example, to the effect that if $u \in L^2[-\pi,\pi]$ and $\vartheta(\lambda)$ is the Fourier transform of u then the map from ϑ to $\{\vartheta(\lambda_k)\}$, with the λ_k constrained as in (5.16), is continuous with respect to the L^2_Γ norm of ϑ on any vertical line Γ and the ℓ^2 norm of $\{\vartheta(\lambda_k)\}$. The relation (5.18) together with the fact that $\varphi(\lambda)/(\varphi'(\lambda_k)(\lambda - \lambda_k))$ is the Fourier transform of $q_k(t)$, together with the result just stated, shows T^* to be bounded. But then $T = (T^*)^*$ is also bounded.

In order to complete the proof of the theorem it is clearly enough to show that T^{-1}, or, equivalently, $(T^*)^{-1} = (T^{-1})^*$ is bounded. That argument proceeds as follows. From

the boundedness of T and the Plancherel theorem, we see that if $\{a_k\} \in \ell^2$ then

$$\psi(\lambda) = \sum_{k \in K} \frac{a_k}{\lambda - \lambda_k} \in L^2[-\pi,\pi] \tag{5.23}$$

and there is a positive constant M_1 such that

$$\|\psi\|^2_{L^2(\Gamma_1)} = \int_{\Gamma_1} |\psi(\lambda)|^2 |d\lambda| \le M \sum_{k \in K} |a_k|^2 . \tag{5.24}$$

Now let $\{b_k\} \in \ell^2$. We want to obtain, in order to bound $(T^*)^{-1}$, a bound, in $L^2[-\pi,\pi]$, on the sum

$$\sum_{k \in K} b_k q_k(t) .$$

Therefore, we consider the corresponding sum of transforms

$$\sum_{k \in K} \frac{b_k}{\varphi'(\lambda_k)} \frac{\varphi(\lambda)}{\lambda - \lambda_k} .$$

A priori we know nothing about the convergence of this sum. But now consider

$$\psi(\lambda) = \sum_{k \in K} \frac{b_k}{\varphi'(\lambda_k)} \frac{1}{\lambda - \lambda_k} .$$

Since the numbers $|\varphi'(\lambda_k)|$ are bounded away from zero and $\{b_k\} \in \ell^2$, we have

$$\{ \frac{b_k}{\varphi'(\lambda_k)} \} \in \ell^2$$

and therefore, replacing the a_k in (5.23), (5.24) by the $b_k/\varphi'(\lambda_k)$,

$$\|\psi\|^2_{L^2(\Gamma_1)} \le M \sum_{k \in K} |\frac{b_k}{\varphi'(\lambda_k)}| \le M_1 \sum_{k \in K} |b_k|^2 ,$$

where

$$M_1 = \frac{M}{\inf_{k \in K} |\varphi'(\lambda_k|^2} > 0 .$$

Let

$$\vartheta(\lambda) = \varphi(\lambda)\psi(\lambda) = \varphi(\lambda)\Big(\sum_{k \in K} \frac{b_k}{\varphi'(\lambda_k)} \frac{1}{\lambda - \lambda_k} \Big) . \tag{5.25}$$

Since $\varphi(\lambda)$ is bounded on Γ_1 ,

$$\|\vartheta\|^2_{L^2(\Gamma_1)} \le M_2 \sum_{k \in K} |b_k|^2 \tag{5.26}$$

for some $M_2 > 0$. From the growth properties of $\varphi(\lambda)$ and $\psi(\lambda)$ we infer that

Proposition 5.6 can be applied to $\vartheta(\lambda)$. Now (5.25) shows that

$$\vartheta(\lambda_k) = b_k$$

so Proposition 5.6 gives

$$\vartheta(\lambda) = \sum_{k \in K} \frac{b_k}{\varphi'(\lambda_k)} \frac{\varphi(\lambda)}{\lambda - \lambda_k}, \tag{5.27}$$

the same result as we would get from (5.25) by taking $\varphi(\lambda)$ inside the summation sign. Formula (5.27) gives the expansion of $\vartheta(\lambda)$ in terms of the functions $\varphi(\lambda)/[\varphi'(\lambda_k)(\lambda - \lambda_k)]$. Formula (5.26) can be used, with partial sums, to establish the convergence of (5.27) in $L^2(\Gamma_1)$ and that formula likewise shows that

$$\|\vartheta\|^2_{L^2(\Gamma_1)} = \Big\| \sum_{k \in K} b_k \frac{\varphi(\lambda)}{\varphi'(\lambda_k)(\lambda - \lambda_k)} \Big\|^2_{L^2(\Gamma_1)} \leq M_2 \sum_{k \in K} |b_k|^2 = M_2 \|\{b_k\}\|^2_{\ell^2}.$$

Applying the Plancherel theorem, there is an $M_3 > 0$ such that

$$\Big\| \sum_{k \in K} b_k q_k \Big\|^2_{L^2[-\pi,\pi]} \leq M_3 \|\{b_k\}\|^2_{\ell^2}$$

and we conclude that $(T^*)^{-1}$, and hence T^{-1}, is bounded, completing the proof of the theorem.

It may be useful to review the outline of the proof above. A more or less standard result shows that T is bounded. The boundedness of $(T^*)^{-1}$ is obtained by introducing two operators. The operator C carries $\{b_k\} \in \ell^2$ into $\{\frac{b_k}{\varphi'(\lambda_k)}\} \in \ell^2$ if the $|\varphi'(\lambda_k)|$ are bounded below. The operator B is the Bary map described at the beginning of this section. What we have shown is that

$$(T^*)^{-1} = B\,T\,C\,.$$

Since B and C are bounded (and boundedly invertible, in fact) this result allows one to infer the boundedness of $(T^*)^{-1}$ from the boundedness of T. The essential contribution of this report lies in the development of the A-Laplace transform, establishing the equivalence of $\|\psi\|_{L^2(\Gamma_1)}$ and $\|x\|_{L^2[-\pi,\pi]}$, where ψ is the A-Laplace transform of x, which permits the boundedness of T to be restated in the form (5.24).

These results can be extended to $H^m[-\pi,\pi]$, replacing (5.17) by

$$|F(\eta,\lambda)| \geq M(1 + |\lambda|)^m \mathrm{dis}(\lambda,\Lambda).$$

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We present a theory relating the completeness and independence properties of sets of complex exponentials $\{e^{\lambda_k t}\}$ in the Sobolev spaces $H^m[a,b]$ to strongly continuous groups of bounded operators on $H^m[a,b]$.

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